Inequivalent Vacuum States in Algebraic Quantum Theory

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Abstract

The Gelfand–Naimark–Sigal representation construction is considered in a general case of topological involutive algebras of quantum systems, including quantum fields, and inequivalent state spaces of these systems are characterized. We aim to show that, from the physical viewpoint, they can be treated as classical fields by analogy with a Higgs vacuum field.

Contents

1 Introduction

No long ago, one thought of vacuum in quantum field theory (QFT) as possessing no physical characteristics, and thus being invariant under any symmetry transformation. It is exemplified by a particleless Fock vacuum (Section 8). Contemporary gauge models of fundamental interactions however have arrived at a concept of the Higgs vacuum (HV). By contrast to the Fock one, HV is equipped with nonzero characteristics, and consequently is non-invariant under transformations.

For instance, HV in Standard Model of particle physics is represented by a constant background classical Higgs field, in fact, inserted by hand into a field Lagrangian, whereas its true physical nature still remains unclear. In particular, somebody treats it as a *sui generis* condensate by analogy with the Cooper one, and its appearance is regarded as a phase transition characterized by some symmetry breakdown [3, 5, 31].

Thus, we come to a concept of different inequivalent and, in particular, non-invariant vacua [53]. Here, we consider some models of these vacua in the framework of algebraic quantum theory (AQT). We aim to show that, from the physical viewpoint, their characteristics are classical just as we observe in a case of the above-mentioned Higgs vacuum.

In AQT, a quantum system is characterized by a topological involutive algebra A and a family of continuous positive forms on A. Elements of A are treated as quantum objects, and we call A the quantum algebra. In this framework, values of positive forms on A are regarded as numerical averages of elements of A. In the spirit of Copenhagen interpretation of quantum theory, one can think of positive forms on A as being classical objects.

A corner stone of AQT is the following Gelfand–Naimark–Segal (GNS) representation theorem [24, 27, 57].

Theorem 1.1. Let A be a unital topological involutive algebra and f a positive continuous form on A such that $f(\mathbf{1}) = 1$ (i.e., f is a state). There exists a strongly cyclic Hermitian representation (π_f, θ_f) of A in a Hilbert space E_f with a cyclic vector θ_f such that

$$f(a) = \langle \pi(a)\theta_\phi | \theta_\phi \rangle, \qquad a \in A. \tag{1.1}$$

□

It should be emphasized that a Hilbert space E_f in Theorem 1.1 is a completion of the quotient of an algebra A with respect to an ideal generated by elements $a \in A$ such that $f(aa^*) = 0$, and the cyclic vector θ_f is the image of the identity $\mathbf{1} \in A$ in this quotient. Thus, a carrier space of a representation of A and its cyclic vector in Theorem 1.1 comes from a quantum algebra A, and they also can be treated as quantum objects.

Since θ_f (1.1) is a cyclic vector, we can think of it as being a vacuum vector and, accordingly, a state f as being a vacuum of an algebra A. Let us note that a vacuum vector θ_f is a quantum

object, whereas a vacuum f of A is the classical one. In particular, a quantum algebra A acts on quantum vectors, but not on its vacua (states).

Since vacua are classical objects, they are parameterized by classical characteristics. A problem is that different vacua of A define inequivalent cyclic representations of a quantum algebra A in general. In this case, they are called inequivalent.

We say that a quantum algebra A performs a transition between its vacua f and f' if there exist elements $b, b' \in A$ such that $f'(a) = f(b^+ab)$ and $f(a) = f'(b'^+ab')$ for all $a \in A$. In this case cyclic representations π_f and $\pi_{f'}$ and, accordingly, vacua f and f' are equivalent (Theorem 6.1). A problem thus is to characterize inequivalent vacua of a quantum system.

One can say something in the following three variants.

(i) If a quantum algebra A is a unital C^*-algebra, Theorem 1.1 comes to well-known GNS (Theorem 2.7), and we have a cyclic representation of A by bounded operators in a Hilbert space (Section 2). This is a case of quantum mechanics.

(ii) A quantum algebra A is a nuclear involutive algebra (Theorem 6.3). In particular, this is just the case of quantum field theory (Sections ?9 and 10).

(iii) Given a group G of automorphisms of a quantum algebra A, its vacuum is f invariant only under a proper subgroup of G. This is the case of spontaneous symmetry breaking in a quantum system (Sections 12 and 13).

If a quantum algebra A is a unital C^*-algebra, one can show that a set $F(A)$ of states of A is a weakly*-closed convex hull of a set $P(A)$ of pure states of A, and it is weakly* compact (Theorem 3.10). A set $P(A)$ of pure states of A, in turn, is a topological bundle over the spectrum $\widehat{A}$ of A whose fibres are projective Hilbert space. The spectrum $\widehat{A}$ of A is a set of its nonequivalent irreducible representations provided with the inverse image of the Jacobson topology. It is quasi-compact.

In accordance with Theorem 3.2 a unital C^*-algebra A of a quantum system performs invertible transitions between different vacua iff they are equivalent. At the same time, one can enlarge an algebra A to some algebra $B(E_F)$ so that all states of A become equivalent states of $B(E_F)$ (Theorem 3.11). Moreover, this algebra contains the superselection operator T (3.7) which belongs to the commutant of A and whose distinct eigenvalues characterize different vacua of A.

In Section 4, an infinite qubit system modelled on an arbitrary set S is studied. Its quantum C^*-algebra A_S possesses pure states whose set is a set of maps σ (4.1) of a set S to the unit sphere in $\mathbb{C}^2$. They are equivalent iff the relation (4.3) is satisfied and, in particular, if maps σ and σ' differ from each other on a finite subset of S. By analogy with a Higgs vacuum, one can treat the maps σ (4.1) as classical vacuum fields.

In Section 5, we consider an example of a locally compact group G and its group algebra $L^1_{\mathbb{C}}(G)$ of equivalence classes of complex integrable functions on G. This is a Banach involutive algebra with an approximate identity. There is one-to-one correspondence between the representations of this algebra and the strongly continuous unitary representations of a group G (Theorem 5.1). Continuous positive forms on $L^1_{\mathbb{C}}(G)$ and, accordingly, its cyclic representations are parameterized by continuous positive-definite functions ψ on G as classical vacuum fields (Theorem 5.3). If ψ is square-integrable, the corresponding cyclic representation of $L^1_{\mathbb{C}}(G)$ is contained in the regular representation (5.8). In this case, distinct square integrable continuous positive-definite functions ψ and ψ' on G define inequivalent irreducible representations if they obey the relations (5.9).

However, this is not the case of unnormed topological $*$-algebras. In order to say something,

we restrict our consideration to nuclear algebras (Section 6 and 7).

This technique is applied to the analysis of inequivalent representations of infinite canonical commutative relations (Section 8) and, in particular, free quantum fields, whose states characterized by different masses are inequivalent.

Section 10 addresses the true functional integral formulation of Euclidean quantum theory (Section 10). These integrals fail to be translationally invariant that enables one to model a Higgs vacuum a translationall inequivalent state (Section 11).

Sections 12 and 13 are devoted to the phenomenon of spontaneous symmetry breaking when a state of a quantum algebra A fails to be stationary only with respect to some some proper subgroup H of a group G of automorphisms of A. Then a set of inequivalent states of these algebra generated by these automorphisms is a subset of the quotient G/H.

In particular, just this fact motivates us to describe classical Higgs fields as sections of a fibre bundle with a typical fibre G/H [52, 53, 54].

2 GNS construction. Bounded operators

We start with a GNS representation of a topological involutive algebra A by bounded operators in a Hilbert space. This is the case of Banach involutive algebras with an approximate identity (Theorem 2.7). Without a loss of generality, we however restrict our consideration to GNS representations of C^*-algebras because any involutive Banach algebra A with an approximate identity defines the enveloping C^*-algebra $A^\dagger$ such that there is one-to-one correspondence between the representations of A and those of $A^\dagger$ (Remark 2.5).

Let us recall the standard terminology [18, 24]. A complex associative algebra A is called involutive (a *-algebra) if it is provided with an involution $*$ such that

$$(a^*)^* = a, \quad (a+\lambda b)^* = a^* + \overline{\lambda} b^*, \quad (ab)^* = b^* a^*, \quad a, b \in A, \quad \lambda \in \mathbb{C}.$$

An element $a \in A$ is normal if $aa^* = a^*a$, and it is Hermitian or self-adjoint if $a^* = a$. If A is a unital algebra, a normal element such that $aa^* = a^*a = \mathbf{1}$ is called the unitary one.

A *-algebra A is called the normed algebra (resp. the Banach algebra) if it is a normed (resp. complete normed) vector space whose norm $\|.\|$ obeys the multiplicative conditions

$$\|ab\| \leq \|a\| \|b\|, \qquad \|a^*\| = \|a\|, \qquad a, b \in A.$$

A Banach *-algebra A is said to be a C^*-algebra if $\|a\|^2 = \|a^*a\|$ for all $a \in A$. If A is a unital C^*-algebra, then $\|\mathbf{1}\| = 1$. A C^*-algebra is provided with a normed topology, i.e., it is a topological *-algebra.

Remark 2.1. It should be emphasized that by a morphism of normed algebras is meant a morphism of the underlying *-algebras, without any condition on the norms and continuity. At the same time, an isomorphism of normed algebras means always an isometric morphism. Any morphism ϕ of C^*-algebras is automatically continuous due to the property

$$\|\phi(a)\| \leq \|a\|, \qquad a \in A. \tag{2.1}$$

$\diamondsuit$

Any *-algebra A can be extended to a unital algebra $\widetilde{A} = \mathbb{C} \oplus A$ by the adjunction of the identity $\mathbf{1}$ to A. The unital extension of A also is a *-algebra with respect to the operation

$$(\lambda\mathbf{1} + a)^* = (\overline{\lambda}\mathbf{1} + a^*), \qquad \lambda \in \mathbb{C}, \quad a \in A.$$

If A is a C^*-algebra, a norm on A is uniquely prolonged to the norm

$$\|\lambda\mathbf{1} + a\| = \sup_{\|a'\|\leq 1} \|\lambda a' + aa'\|$$

on $\widetilde{A}$ which makes $\widetilde{A}$ a C^*-algebra.

One says that a Banach algebra A admits an approximate identity if there is a family $\{u_\iota\}_{\iota\in I}$ of elements of A, indexed by a directed set I, which possesses the following properties:

- $\|u_\iota\| < 1$ for all $\iota \in I$,
- $\|u_\iota a - a\| \to 0$ and $\|au_\iota - a\| \to 0$ for every $a \in A$.

It should be noted that the existence of an approximate identity is an essential condition for many results (see, e.g., Theorems 2.6 and 2.7).

For instance, a C^*-algebra has an approximate identity. Conversely, any Banach *-algebra A with an approximate identity admits the enveloping C^*-algebra $A^\dagger$ (Remark 2.5) [18, 24].

An important example of C^*-algebras is an algebra $B(E)$ of bounded (and, equivalently, continuous) operators in a Hilbert space E (Section 14.2). Every closed *-subalgebra of $B(E)$ is a C^*-algebra and, conversely, every C^*-algebra is isomorphic to a C^*-algebra of this type (Theorem 2.1).

An algebra $B(E)$ is endowed with the operator norm

$$\|a\| = \sup_{\|e\|_E=1} \|ae\|_E, \qquad a \in B(E). \tag{2.2}$$

This norm brings the *-algebra $B(E)$ of bounded operators in a Hilbert space E into a C^*-algebra. The corresponding topology on $B(E)$ is called the normed operator topology.

One also provides $B(E)$ with the strong and weak operator topologies, defined by the families of seminorms

$$\{p_e(a) = \|ae\|, \quad e \in E\},$$
$$\{p_{e,e'}(a) = |\langle ae|e'\rangle|, \quad e, e' \in E\},$$

respectively. The normed operator topology is finer than the strong one which, in turn, is finer than the weak operator topology. The strong and weak operator topologies on a subgroup $U(E) \subset B(E)$ of unitary operators coincide with each other.

It should be emphasized that $B(E)$ fails to be a topological algebra with respect to strong and weak operator topologies. Nevertheless, the involution in $B(E)$ also is continuous with respect to the weak operator topology, while the operations

$$B(E) \ni a \to aa' \in B(E), \qquad B(E) \ni a \to a'a \in B(E),$$

where a' is a fixed element of $B(E)$, are continuous with respect to all the above mentioned operator topologies.

Remark 2.2. Let N be a subset of $B(E)$. The commutant N' of N is a set of elements of $B(E)$ which commute with all elements of N. It is a subalgebra of $B(E)$. Let $N'' = (N')'$ denote the

bicommutant. Clearly, $N \subset N''$. A *-subalgebra B of $B(E)$ is called the von Neumann algebra if $B = B''$. This property holds iff B is strongly (or, equivalently, weakly) closed in $B(E)$ [18]. For instance, $B(E)$ is a von Neumann algebra. Since a strongly (weakly) closed subalgebra of $B(E)$ also is closed with respect to the normed operator topology on $B(E)$, any von Neumann algebra is a C^*-algebra. $\diamondsuit$

Remark 2.3. A bounded operator in a Hilbert space E is called completely continuous if it is compact, i.e., it sends any bounded set into a set whose closure is compact. An operator $a \in B(E)$ is completely continuous iff it can be represented by the series

$$a(e) = \sum_{k=1}^{\infty} \lambda_k \langle e|e_k \rangle e_k, \tag{2.3}$$

where e_k are elements of a basis for E and λ_k are positive numbers which tend to zero as $k \to \infty$. For instance, every degenerate operator (i.e., an operator of finite rank which sends E onto its finite-dimensional subspace) is completely continuous. A completely continuous operator a is called the Hilbert–Schmidt operator if the series

$$\|a\|^2_{\rm HS} = \sum_k \lambda_k^2$$

converges. Hilbert–Schmidt operators make up an involutive Banach algebra with respect to this norm, and it is a two-sided ideal of an algebra $B(E)$. A completely continuous operator a in a Hilbert space E is called a nuclear operator if the series

$$||a||_{\rm Tr} = \sum_k \lambda_k$$

converges. Nuclear operators make up an involutive Banach algebra with respect to this norm, and it is a two-sided ideal of an algebra $B(E)$. Any nuclear operator is the Hilbert–Schmidt one. Moreover, the product of arbitrary two Hilbert–Schmidt operators is a nuclear operator, and every nuclear operator is of this type. $\diamondsuit$

Let us consider representations of *-algebras by bounded operators in Hilbert spaces [18, 39]. It is a morphism π of a *-algebra A to an algebra $B(E)$ of bounded operators in a Hilbert space E, called the carrier space of π. Representations throughout are assumed to be non-degenerate, i.e., there is no element $e \neq 0$ of E such that $Ae = 0$ or, equivalently, AE is dense in E.

Theorem 2.1. If A is a C^*-algebra, there exists its exact (isomorphic) representation. $\Box$

Theorem 2.2. A representation π of a *-algebra A is uniquely prolonged to a representation $\widetilde{\pi}$ of the unital extension $\widetilde{A}$ of A. $\Box$

Let $\{\pi^\iota\}$, $\iota \in I$, be a family of representations of a *-algebra A in Hilbert spaces E^ι. If the set of numbers $\|\pi^\iota(a)\|$ is bounded for each $a \in A$, one can construct a bounded operator $\pi(a)$ in a Hilbert sum $\oplus E^\iota$ whose restriction to each E^ι is $\pi^\iota(a)$.

Theorem 2.3. This is the case of a C^*-algebra A due to the property (2.1). Then π is a representation of A in $\oplus E^\iota$, called the Hilbert sum

$$\pi = \bigoplus_I \pi^\iota \tag{2.4}$$

of representations π^ι. $\square$

Given a representation π of a *-algebra A in a Hilbert space E, an element $\theta \in E$ is said to be a cyclic vector for π if the closure of $\pi(A)\theta$ is equal to E. Accordingly, π (or a more strictly a pair (π, θ)) is called the cyclic representation.

Theorem 2.4. Every representation of a *-algebra A is a Hilbert sum of cyclic representations. $\square$

Remark 2.4. It should be emphasized, that given a cyclic representation (π, θ) of a *-algebra A in a Hilbert space E, a different element θ' of E is a cyclic for π iff there exist some elements $b, b' \in A$ such that $\theta' = \pi(b)\theta$ and $\theta = \pi(b')\theta'$. $\diamondsuit$

Let A be a *-algebra, π its representation in a Hilbert space E, and θ an element of E. Then a map

$$\omega_\theta : a \to \langle \pi(a)\theta|\theta\rangle \tag{2.5}$$

is a positive form on A. It is called the vector form defined by π and θ.

Therefore, let us consider positive forms on a *-algebra A. Given a positive form f, a Hermitian form

$$\langle a|b\rangle = f(b^*a), \qquad a, b \in A, \tag{2.6}$$

makes A a pre-Hilbert space. If A is a normed *-algebra, continuous positive forms on A are provided with a norm

$$\|f\| = \sup_{\|a\|=1} |f(a)|, \qquad a \in A. \tag{2.7}$$

Theorem 2.5. Let A be a unital Banach *-algebra such that $\|\mathbf{1}\| = 1$. Then any positive form on A is continuous. $\square$

In particular, positive forms on a C^*-algebra always are continuous. Conversely, a continuous form f on an unital C^*-algebra is positive iff $f(\mathbf{1}) = \|f\|$. It follows from this equality that positive forms on a unital C^*-algebra A obey a relation

$$\|f_1 + f_2\| = \|f_1\| + \|f_2\|. \tag{2.8}$$

Let us note that a continuous positive form on a topological *-algebra A admits different prolongations onto the unital extension $\widetilde{A}$ of A. Such a prolongation is unique in the following case [18].

Theorem 2.6. Let f be a positive form on a Banach *-algebra A with an approximate identity. It is extended to a unique positive form $\widetilde{f}$ on the unital extension $\widetilde{A}$ of A such that $\widetilde{f}(\mathbf{1}) = \|f\|$. $\square$

A key point is that any positive form on a C^*-algebra equals a vector form defined by some cyclic representation of A in accordance with the following GNS representation construction [18, 24].

Theorem 2.7. Let f be a positive form on a Banach *-algebra A with an approximate identity and $\widetilde{f}$ its continuous positive prolongation onto the unital extension $\widetilde{A}$ (Theorems 2.5 and 2.6). Let N_f be a left ideal of $\widetilde{A}$ consisting of those elements $a \in A$ such that $\widetilde{f}(a^*a) = 0$. The quotient $\widetilde{A}/N_f$ is a Hausdorff pre-Hilbert space with respect to the Hermitian form obtained from $\widetilde{f}(b^*a)$

(2.6) by passage to the quotient. We abbreviate with E_f the completion of $\widetilde{A}/N_f$ and with θ_f the canonical image of $\mathbf{1} \in \widetilde{A}$ in $\widetilde{A}/N_f \subset E_f$. For each $a \in \widetilde{A}$, let $\tau(a)$ be an operator in $\widetilde{A}/N_f$ obtained from the left multiplication by a in $\widetilde{A}$ by passage to the quotient. Then the following holds.

(i) Each $\tau(a)$ has a unique extension to a bounded operator $\pi_f(a)$ in a Hilbert space E_f.

(ii) A map $a \to \pi_f(a)$ is a representation of A in E_f.

(iii) A representation π_f admits a cyclic vector θ_f.

(iv) $f(a) = \langle \pi(a)\theta_f|\theta_f\rangle$ for each $a \in A$. □

The representation π_f and the cyclic vector θ_f in Theorem 2.7 are said to be defined by a form f, and a form f equals the vector form defined by π_f and θ_f.

As was mentioned above, we further restrict our consideration of the GNS construction in Theorem 2.7 to unital C^*-algebras in view of the following [18, 24].

Remark 2.5. Let A be an involutive Banach algebra A with an approximate identity, and let $P(A)$ be the set of pure states of A (Remark 3.2). For each $a \in A$, we put

$$\|a\|' = \sup_{f\in P(A)} f(aa^*)^{1/2}, \qquad a \in A. \tag{2.9}$$

It is a seminorm on A such that $\|a\|' \leq \|a\|$. If A is a C^*-algebra, $\|a\|' = \|a\|$ due to the relation (2.1) and the existence of an isomorphic representation of A. Let $\mathcal{I}$ denote the kernel of $\|.\|'$. It consists of $a \in A$ such that $\|a\|' = 0$. Then the completion $A^\dagger$ of the factor algebra $A/\mathcal{I}$ with respect to the quotient of the seminorm (2.9), is a C^*-algebra, called the enveloping C^*-algebra of A. There is the canonical morphism $\tau : A \to A^\dagger$. Clearly, $A = A^\dagger$ if A is a C^*-algebra. The enveloping C^*-algebra $A^\dagger$ possesses the following important properties.

• If π is a representation of A, there is exactly one representation $\pi^\dagger$ of $A^\dagger$ such that $\pi = \pi^\dagger \circ \tau$. Moreover, the map $\pi \to \pi^\dagger$ is a bijection of a set of representations of A onto a set of representations of $A^\dagger$.

• If f is a continuous positive form on A, there exists exactly one positive form $f^\dagger$ on $A^\dagger$ such that $f = f^\dagger \circ \tau$. Moreover, $\|f^\dagger\| = \|f\|$. The map $f \to f^\dagger$ is a bijection of a set of continuous positive forms on A onto a set of positive forms on $A^\dagger$. ♢

Moreover, the cyclic vector θ_f in Theorem 2.7 defined by a positive form f is the image of the identity under the quotient $\widetilde{A} \to \widetilde{A}/N_f$, and thus the GNS construction necessarily is concerned with unital algebras. In view of Theorems 2.2 and 2.6, we therefore can restrict our consideration to unital C^*-algebras.

3 Inequivalent vacua

Let A be a unital C^*-algebra of a quantum systems. As was mentioned above, positive forms on a C^* algebra are said to be equivalent if they define its equivalent cyclic representations.

Remark 3.1. Let us recall that two representations π_1 and π_2 of a *-algebra A in Hilbert spaces E_1 and E_2 are equivalent if there is an isomorphism $\gamma : E_1 \to E_2$ such that

$$\pi_2(a) = \gamma \circ \pi_1(a) \circ \gamma^{-1}, \qquad a \in A. \tag{3.1}$$

In particular, if representations are equivalent, their kernels coincide with each other. ♢

Given two positive forms f_1 and f_2 on a unital C^*-algebra A, we meet the following three variants.

(i) If $f_1 = f_2$, there is an isomorphism γ of the corresponding Hilbert spaces $\gamma : E_1 \to E_2$ such that the relation (3.1) holds, and moreover

$$\theta_2 = \gamma(\theta_1). \tag{3.2}$$

(ii) Let positive forms f_1 and f_2 be equivalent, but different. Then their equivalence morphism γ fails to obey the relation (3.2).

(iii) Positive forms f_1 and f_2 on A are inequivalent.

In particular, let π be a representation of A in a Hilbert space E, and let θ be an element of E which defines the vector form ω_θ (2.5) on A. Then a representation π contains a summand which is equivalent to the cyclic representation π_{ω_θ} of A defined by a vector form ω_θ.

There are the following criteria of equivalence of positive forms.

Theorem 3.1. Positive forms on a unital C^*-algebra are equivalent only if their kernels contain a common largest closed two-sided ideal. □

Proof. The result follows from the fact that the kernel of a cyclic representation defined by a positive form on a unital C^*-algebra is a largest closed two-sided ideal of the kernel of this form [18] ♢

Theorem 3.2. Positive forms f and f' on a unital C^*-algebra A are equivalent iff there exist elements $b, b' \in A$ such that

$$f'(a) = f(b^+ab), \qquad f(a) = f'(b'^+ab'), \qquad a \in A.$$

□

Proof. Let a positive form f define a cyclic representation (π_f, θ_f) of A in E_f. Let us consider an element $\pi_f(b)\theta_f \in E_f$. In accordance with Remark 2.4, this element is a cyclic element for a representation π_f. It provides a positive form $\omega_{\pi_f(b)\theta_f}$ on A such that $\omega_{\pi_f(b)\theta_f} = f'$. Then a positive form f' defines a cyclic representation $(\pi_{f'}, \theta_{f'})$ of A in a Hilbert space $E_{f'}$ which is isomorphic as $\gamma : E_{f'} \to E_f$ to a cyclic representation $(\pi_f, \pi_f(b)\theta_f)$ in E_f such that the relation (3.2) holds. Conversely, let positive forms f and f' be equivalent. Then a positive form f' defines an isomorphic cyclic representation (π_f, θ') in E_f, but with a different cyclic vector θ'. Then the result follows from Remark 2.4. ♢

In particular, it follows from Theorem 3.2 that given a positive form f on A, the state $f(\mathbf{1})^{-1}f$ of A is equivalent to f. Speaking on equivalent positive forms on A, we therefore can restrict our consideration to states.

For instance, any cyclic representation of a C^*-algebra A is a summand of the Hilbert sum (2.4):

$$\pi_F = \bigoplus_{F(A)} \pi_f, \tag{3.3}$$

of cyclic representations of A where f runs through a set $F(A)$ of all states of A. Since for any element $a \in A$ there exists a state f such that $f(a) \neq 0$, the representation π_F is injective and, consequently, isometric and isomorphic.

A space of continuous forms on a C^*-algebra A is the (topological) dual A' of a Banach space A. It can be provided both with a normed topology defined by the norm (2.7) and a weak*

topology (Section 14.1). It follows from the relation (2.8), that a subset $F(A) \subset A'$ of states is convex and its extreme points are pure states.

Remark 3.2. Let us recall that a positive form f' on a *-algebra A is said to be dominated by a positive form f if $f - f'$ is a positive form [24, 18]. A non-zero positive form f on a *-algebra A is called pure if every positive form f' on A which is dominated by f reads λf, $0 \leq \lambda \leq 1$. $\diamondsuit$

A key point is the following [18]

Theorem 3.3. The cyclic representation of π_f of a C^*-algebra A defined by a positive form f on A is irreducible iff f is a pure form [18] $\square$

In particular, any vector form defined by a vector of a carrier Hilbert space of an irreducible representation is a pure form.

Remark 3.3. Let us note that a representation π of a *-algebra A in a Hilbert space E is called topologically irreducible if the following equivalent conditions hold:
- the only closed subspaces of E invariant under $\pi(A)$ are 0 and E;
- the commutant of $\pi(A)$ in $B(E)$ is a set of scalar operators;
- every non-zero element of E is a cyclic vector for π.

At the same time, irreducibility of π in the algebraic sense means that the only subspaces of E invariant under $\pi(A)$ are 0 and E. If A is a C^*-algebra, the notions of topologically and algebraically irreducible representations are equivalent. It should be emphasized that a representation of a C^*-algebra need not be a Hilbert sum of the irreducible ones. $\diamondsuit$

An algebraically irreducible representation π of a *-algebra A is characterized by its kernel $\operatorname{Ker}\pi \subset A$. This is a two-sided ideal, called primitive. Certainly, algebraically irreducible representations with different kernels are inequivalent, whereas equivalent irreducible representations possesses the same kernel. Thus, we have a surjection

$$\widehat{A} \ni \pi \to \operatorname{Ker}\pi \in \operatorname{Prim}(A) \tag{3.4}$$

of a set $\widehat{A}$ of equivalence classes of algebraically irreducible representations of a *-algebra A onto a set $\operatorname{Prim}(A)$ of primitive ideals of A.

A set $\operatorname{Prim}(A)$ is equipped with the so called Jacobson topology [18]. This topology is not Hausdorff, but obeys the Fréchet axiom, i.e., for any two distinct points of $\operatorname{Prim}(A)$, there is a neighborhood of one of them which does not contain the other. Then a set $\widehat{A}$ is endowed with the coarsest topology such that the surjection (3.4) is continuous. Provided with this topology, $\widehat{A}$ is called the spectrum of a *-algebra A. In particular, one can show the following.

Theorem 3.4. If a *-algebra A is unital, its spectrum $\widehat{A}$ is quasi-compact, i.e., it satisfies the Borel–Lebesgue axiom, but need not be Hausdorff. $\square$

Theorem 3.5. The spectrum $\widehat{A}$ of a C^*-algebra A is a locally quasi-compact space. $\square$

It follows from Theorems 3.4 and 3.5 that the spectrum of a unital C^*-algebra is quasi-compact.

Example 3.4. A C^*-algebra is said to be elementary if it is isomorphic to an algebra $T(E) \subset B(E)$ of compact operators in some Hilbert space E (Example 2.3). Every non-trivial irreducible representation of an elementary C^* algebra $A \cong T(E)$ is equivalent to its isomorphic

representation by compact operators in E [18]. Hence, the spectrum of an elementary algebra is a singleton set. □

By analogy with Theorem 3.2, one can state the following relations between equivalent pure states of a C^*-algebra.

Theorem 3.6. Pure states f and f' of a unital C^*-algebra A are equivalent iff there exists a unitary element $U \in A$ such that the relation

$$f'(a) = f(U^* a U), \qquad a \in A. \tag{3.5}$$

holds. □

Proof. A key point is that, if f is a pure state of a unital C^*-algebra, a pseudo-Hilbert space A/N_f in Theorem 2.7 is complete, i.e., $E_f = A/N_f$. ♢

Corollary 3.7. Let π be an irreducible representation of a unital C^*-algebra A in a Hilbert space E. Given two distinct elements θ_1 and θ_2 of E (they are cyclic for π), the vector forms on A defined by (π, θ_1) and (π, θ_2) are equal iff there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $\theta_1 = \lambda\theta_2$. □

Corollary 3.8. There is one-to-one correspondence between the pure states of a unital C^*-algebra A associated to the same irreducible representation π of A in a Hilbert space E and the one-dimensional complex subspaces of E, i.e, these pure states constitute a projective Hilbert space PE. □

There is an additional important criterion of equivalence of pure states of a unital C^*-algebra [23].

Theorem 3.9. Pure states f and f' of a unital C^*-algebra are equivalent if $\|f - f'\| < 2$. □

Let $P(A)$ denote a set of pure states of a unital C^*-algebra A. Theorem 3.6 implies a surjection $P(A) \to \widehat{A}$. One can show that, if $P(A) \subset A'$ is provided with a relative weak* topology, this surjection is continuous and open, i.e., it is a topological fibre bundle whose fibres are projective Hilbert spaces [18].

Turning to a set $F(A)$ of states of a unital algebra C^*-algebra A, we have the following.

Theorem 3.10. A set $F(A)$ is a weakly*-closed convex hull of a set $P(A)$ of pure states of A. It is weakly* compact [18]. □

Herewith, by virtue of Theorem 3.10, any set of mutually inequivalent pure states of a unital C^*-algebra is totally disconnected in a normed topology, i.e., its connected components are points only.

By virtue of Theorem 3.2, elements of a quantum algebra A can not perform invertible transitions between its inequivalent states. At the same time, one can show the following.

Theorem 3.11. There exists a wider unital C^*-algebra such that inequivalent states of A become its equivalent ones. □

Proof. Let us consider the Hilbert sum π_F (3.3) of cyclic representations of A whose carrier space is a Hilbert sum

$$E_F = \bigoplus_{F(A)} E_f. \tag{3.6}$$

Let $B(E_F)$ be a unital C^* algebra of bounded operators in E_F (3.6). Since the representation π_F of A is exact, an algebra A is isomorphic to a subalgebra of $B(E_F)$. Any state f of A is equivalent to a vector state ω_{θ_f} of $\pi_f(A)$ which also is that of $B(E_F)$. Since all vector states of $B(E_F)$ are equivalent, all states of A are equivalent as those of $B(E_F)$. ♢

An algebra $B(E_F)$ contains the projectors P_f onto summands E_f of E_F (3.6). Let $r(f)$ be some real function on $F(A)$. Then there exists a bounded operator in E_F (3.6), which we denote

$$T = \sum_{F(A)} r(f)P_f, \tag{3.7}$$

such that its restriction to each summand E_f of E_F is $r(f)P_f$. Certainly, this operator belongs to the commutant $\pi_F(A)'$ of $\pi_F(A)$ in $B(E_F)$.

One can think of T (3.7) as being a superselection operator of a quantum system which distinguish its states [27].

4 Example. Infinite qubit systems

Let Q be a two-dimensional complex space $\mathbb{C}^2$ equipped with the standard positive non-degenerate Hermitian form $\langle .|.\rangle_2$. Let M_2 be an algebra of complex 2×2-matrices seen as a C^*-algebra. A system of m qubits is usually described by a Hilbert space $E_m = \overset{m}{\otimes} Q$ and a C^*-algebra $A_m = \overset{m}{\otimes} M_2$, which coincides with the algebra $B(E_m)$ of bounded operators in E_m [32]. One can straightforwardly generalize this description to an infinite set S of qubits by analogy with a spin lattice [20, 24, 50]. Its algebra A_S admits inequivalent irreducible representations.

We follow the construction of infinite tensor products of Hilbert spaces and C^*-algebras in [20]. Let $\{Q_s, s\in S\}$ be a set of two-dimensional Hilbert spaces $Q_s = \mathbb{C}^2$. Let $\underset{S}{\times} Q_s$ be a complex vector space whose elements are finite linear combinations of elements $\{q_s\}$ of the Cartesian product $\prod\limits_S Q_s$ of the sets Q_s. A tensor product $\underset{S}{\otimes} Q_s$ of complex vector spaces Q_s is the quotient of $\underset{S}{\times} Q_s$ with respect to a vector subspace generated by elements of the form:

- $\{q_s\} + \{q'_s\} - \{q''_s\}$, where $q_r + q'_r = q''_r$ for some element $r\in S$ and $q_s = q'_s = q''_s$ for all the others,
- $\{q_s\} - \lambda\{q'_s\}$, $\lambda\in\mathbb{C}$, where $q_r = \lambda q'_r$ for some element $r\in S$ and $q_s = q'_s$ for all the others.

Given a map

$$\sigma : S \to Q, \qquad , \langle\sigma(s)|\sigma(s)\rangle_2 = 1, \tag{4.1}$$

let us consider an element

$$\theta_\sigma = \{\theta_s = \sigma(s)\} \in \prod_S Q_s. \tag{4.2}$$

Let us denote $\otimes^\sigma Q_s$ the subspace of $\underset{S}{\otimes} Q_s$ spanned by vectors $\otimes q_s$ where $q_s \neq \theta_s$ only for a finite number of elements $s\in S$. It is called the θ_σ-tensor product of vector spaces Q_s, $s\in S$. Then $\otimes^\sigma Q_s$ is a pre-Hilbert space with respect to a positive non-degenerate Hermitian form

$$\langle \otimes^\sigma q_s| \otimes^\sigma q'_s\rangle = \prod_{s\in S}\langle q_s|q'_s\rangle_2.$$

Its completion Q^σ_S is a Hilbert space whose orthonormal basis consists of the elements $e_{ir} = \otimes q_s$, $r\in S$, $i = 1, 2$, such that $q_{s\neq r} = \theta_s$ and $q_r = e_i$, where $\{e_i\}$ is an orthonormal basis for Q.

Let now $\{A_s, s \in S\}$ be a set of unital C^*-algebras $A_s = M_2$. These algebras are provided with the operator norm

$$\|a\| = (\lambda_0\overline{\lambda}_0 + \lambda_1\overline{\lambda}_1 + \lambda_2\overline{\lambda}_2 + \lambda_3\overline{\lambda}_3)^{1/2}, \qquad a = i\lambda_0\mathbf{1} + \sum_{i=1,2,3} \lambda_i \tau^i,$$

where τ^i are the Pauli matrices. Given the family $\{\mathbf{1}_s\}$, let us construct the $\{\mathbf{1}_s\}$-tensor product $\otimes A_s$ of vector spaces A_s. One can regard its elements as tensor products of elements of $a_s \in A_s$, $s \in K$, for finite subsets K of S and of the identities $\mathbf{1}_s$, $s \in S \setminus K$. It is easily justified that $\otimes A_s$ is a normed *-algebra with respect to the operations

$$(\otimes a_s)(\otimes a'_s) = \otimes(a_s a'_s), \qquad (\otimes a_s)^* = \otimes a^*_s$$

and a norm

$$\| \otimes a_s\| = \prod_s \|a_s\|.$$

Its completion A_S is a C^*-algebra treated as a quantum algebra of a qubit system modelled over a set S. Then the following holds [20].

Theorem 4.1. Given the element $\theta_\sigma = \{\theta_s\}$ (4.2), the natural representation π^σ of a *-algebra $\otimes A_s$ in the pre-Hilbert space $\otimes^\sigma Q_s$ is extended to an irreducible representation of a C^*-algebra A_S in the Hilbert space Q^σ_S such that $\pi^\sigma(A_S) = B(Q^\sigma_S)$ is an algebra of all bounded operators in Q^σ_S. Conversely, all irreducible representations of A_S are of this type. □

An element $\theta_\sigma \in Q^\sigma_S$ in Theorem 4.1 defines a pure state f_σ of an algebra A_S. Consequently, a set of pure states of this algebra is a set of maps σ (4.1).

Theorem 4.2. Pure states f_σ and $f_{\sigma'}$ of an algebra A_S are equivalent iff

$$\sum_{s\in S} ||\langle\sigma(s)|\sigma'(s)\rangle_2| - 1| < \infty. \tag{4.3}$$

□

In particular, the relation (4.3) holds if maps σ and σ' differ from each other on a finite subset of S.

By analogy with a Higgs vacuum, one can treat the maps σ (4.1) as classical vacuum fields.

5 Example. Locally compact groups

Let G be a locally compact group provided with a Haar measure (Section 14.4). A space $L^1_{\mathbb{C}}(G)$ of equivalence classes of complex integrable functions (or, simply, complex integrable functions) on G is an involutive Banach algebra (Section 14.3) with an approximate identity. As was mentioned above, there is one-to-one correspondence between the representations of this algebra and the strongly continuous unitary representations of a group G (Theorem 5.1). Thus, one can employ the GNS construction in order to describe these representations of G [18, 24].

Let a left Haar measure dg on G hold fixed, and by an integrability condition throughout is meant the dg-integrability.

A uniformly (resp. strongly) continuous unitary representation of a locally compact group G in a Hilbert space E is a continuous homeomorphism π of G to a subgroup $U(E) \subset B(E)$ of

unitary operators in E provided with the normed (resp. strong) operator topology. A uniformly continuous representation is strongly continuous. However, the uniform continuity of a representation is rather rigorous condition. For instance, a uniformly continuous irreducible unitary representation of a connected locally compact real Lie group is necessarily finite-dimensional. Therefore, one usually studies strongly continuous representations of locally compact groups.

In this case, any element ξ of a carrier Hilbert space E yields the continuous map $G \ni g \to \pi(g)\xi \in E$. Since strong and weak operator topologies on a unitary group $U(E)$ coincide, we have a bounded continuous complex function

$$\varphi_{\xi,\eta}(g) = \langle \pi(g)\xi | \eta \rangle \tag{5.1}$$

on G for any fixed elements $\xi, \eta \in E$. It is called the coefficient of a representation π. There is an obvious equality

$$\varphi_{\xi,\eta}(g) = \overline{\varphi_{\eta,\xi}(g^{-1})}.$$

The Banach space $L^1_{\mathbb{C}}(G)$ of integrable complex functions on G is provided with the structure of an involutive Banach algebra with respect to the contraction $f_1 * f_2$ (14.25) and the involution

$$f(g) \to f^*(g) = \Delta(g^{-1})\overline{f(g^{-1})},$$

where Δ is the modular function of G. It is called the group algebra of G. A map $f \to f(g)dg$ defines an isometric monomorphism of $L^1_{\mathbb{C}}(G)$ to a Banach algebra $M^1(G, \mathbb{C})$ of bounded complex measures on G provided with the involution $\mu^* = \overline{\mu^{-1}}$. Unless otherwise stated, $L^1_{\mathbb{C}}(G)$ will be identified with its image in $M^1(G, \mathbb{C})$. In particular, a group algebra $L^1_{\mathbb{C}}(G)$ admits an approximate identity which converges to the Dirac measure $\varepsilon_{\mathbf{1}} \in M^1(G, \mathbb{C})$.

Remark 5.1. The group algebra $L^1_{\mathbb{C}}(G)$ is not a C^*-algebra. Its enveloping C^*-algebra $C^*(G)$ is called the C^*-algebra of a locally compact group G. ♢

Unitary representations of a locally compact group G and representations of a group algebra $L^1_{\mathbb{C}}(G)$ are related as follows [18].

Theorem 5.1. There is one-to-one correspondence between the (strongly continuous) unitary representations π of a locally compact group G and the representations π^L (5.3) of its group algebra $L^1_{\mathbb{C}}(G)$. □

Proof. Let π be a (strongly continuous) unitary representation of G in a Hilbert space E. Given a bounded positive measure μ on G, let us consider the integrals

$$\varphi_{\xi,\eta}(\mu) = \int \langle \pi(g)\xi | \eta \rangle \mu$$

of the coefficient functions $\varphi_{\xi,\eta}(g)$ (5.1) for all $\xi, \eta \in E$. There exists a bounded operator $\pi(\mu) \in B(E)$ in E such that

$$\langle \pi(\mu)\xi | \eta \rangle = \varphi_{\xi,\eta}(\mu), \qquad \xi, \eta \in E.$$

It is called the operator-valued integral of $\pi(g)$ with respect to the measure μ, and is denoted by

$$\pi(\mu) = \int \pi(g)\mu(g). \tag{5.2}$$

The assignment $\mu \to \pi(\mu)$ provides a representation of a Banach *-algebra $M^1(G,\mathbb{C})$ in E. Its restriction

$$\pi^L(f) = \int \pi(g)f(g)dg \in B(E) \tag{5.3}$$

to $L^1_{\mathbb{C}}(G)$ is non-degenerate. One says that the representations (5.2) of $M^1(G,\mathbb{C})$ and (5.3) of $L^1_{\mathbb{C}}(G)$ are determined by a unitary representation π of G. Conversely, let π^L be a representation of a Banach *-algebra $L^1_{\mathbb{C}}(G)$ in a Hilbert space E. There is a monomorphism $g \to \varepsilon_g$ of a group G onto a subgroup of Dirac measures ε_g, $g \in G$, of an algebra $M^1(G,\mathbb{C})$. Let $\{u_\iota(q)\}_{\iota\in I}$ be an approximate identity in $L^1_{\mathbb{C}}(G)$. Then $\{\pi^L(u_\iota)\}$ converges to an element of $B(E)$ which can be seen as a representation $\pi^L(\varepsilon_{\mathbf{1}})$ of the unit element $\varepsilon_{\mathbf{1}}$ of $M^1(G,\mathbb{C})$. Accordingly, $\{\pi^L(\gamma(g)u_\iota)\}$ converges to $\pi^L(\varepsilon_g)$. Thereby, we obtain the (strongly continuous) unitary representation $\pi(g) = \pi^L(\varepsilon_g)$ of a group G in a Hilbert space E. Moreover, the representation (5.3) of $L^1_{\mathbb{C}}(G)$ determined by this representation π of G coincides with the original representation π^L of $L^1_{\mathbb{C}}(G)$. $\diamondsuit$

Moreover, π and π^L have the same cyclic vectors and closed invariant subspaces. In particular, a representation π^L of $L^1_{\mathbb{C}}(G)$ is topologically irreducible iff the associated representation π of G is so. It should be emphasized that, since $L^1_{\mathbb{C}}(G)$ is not a C^*-algebra, its topologically irreducible representations need not be algebraically irreducible. By irreducible representations of a group G, we will mean only its topologically irreducible representations.

Theorem 5.1 enables us to apply the GNS construction (Theorem 2.7) in order to characterize unitary representations of G by means of positive continuous forms on $L^1_{\mathbb{C}}(G)$.

In accordance with Remark 14.7, a continuous form on a group algebra $L^1_{\mathbb{C}}(G)$ is defined as

$$\phi(f) = \int \psi(g)f(g)dg \tag{5.4}$$

by an element ψ of a space $L^\infty_{\mathbb{C}}(G)$ of infinite integrable complex functions on G (Remark 14.7). However, a function ψ should satisfy the following additional condition in order that the form (5.4) to be positive.

A continuous complex function ψ on G is called positive-definite if

$$\sum_{i,j} \psi(g_j^{-1}g_i)\overline{\lambda}_i\lambda_j \geq 0$$

for any finite set $g_1,\ldots,g_m$ of elements of G and any complex numbers $\lambda_1,\ldots,\lambda_m$. In particular, if $m=2$ and $g_1 = \mathbf{1}$, we obtain

$$\psi(g^{-1}) = \overline{\psi(g)}, \qquad |\psi|(g) \leq \psi(\mathbf{1}), \qquad g \in G,$$

i.e., $\psi(\mathbf{1})$ is bounded.

Lemma 5.2. The continuous form (5.4) on $L^1_{\mathbb{C}}(G)$ is positive iff $\psi \in L^\infty_{\mathbb{C}}(G)$ locally almost everywhere equals a continuous positive-definite function. $\square$

Then cyclic representations of a group algebra $L^1_{\mathbb{C}}(G)$ and the unitary cyclic representations of a locally compact group G are defined by continuous positive-definite functions on G in accordance with the following theorem.

Theorem 5.3. Let π_ψ be a representation of $L^1_{\mathbb{C}}(G)$ in a Hilbert space E_ψ and θ_ψ a cyclic vector for π_ψ which are determined by the form (5.4). Then the associated unitary representation π_ψ of G in E_ψ is characterized by a relation

$$\psi(g) = \langle \pi(g)\theta_\psi|\theta_\psi\rangle. \tag{5.5}$$

Conversely, a complex function ψ on G is continuous positive-definite iff there exists a unitary representation π_ψ of G and a cyclic vector θ_ψ for π_ψ such that the equality (5.5) holds. $\Box$

By analogy with a Higgs vacuum, one can think of functions ψ in Theorem 5.3 as being classical vacuum fields.

Example 5.2. Let a group G acts on a Hausdorff topological space Z on the left. Let μ be a quasi-invariant measure on Z under a transformation group G, i.e., $\gamma(g)\mu = h_g\mu$ where h_g is the Radon–Nikodym derivative in Theorem 14.1. Then there is a representation

$$G \ni g : f \to \Pi(g)f, \qquad (\Pi(g)f)(z) = h_g^{1/2}(z)f(gz) \tag{5.6}$$

of G in a Hilbert space $L^2_{\mathbb{C}}(Z,\mu)$ of square μ-integrable complex functions on Z [17]. It is a unitary representation due to the equality

$$\|f\|_\mu = \int |f(z)|\mu(z) = \int |f(g(z)|^2\mu(g(z)) = \int h_g(z)|f(g(z)|^2\mu(z) = \|\Pi(g)f\|^2_\mu.$$

A group G can be equipped with the coarsest topology such that the representation (5.6) is strongly continuous. For instance, let $Z = G$ be a locally compact group, and let $\mu = dg$ be a left Haar measure. Then the representation (5.6) comes to the left-regular representation

$$(\Pi(g)f)(q) = f(g^{-1}q), \qquad f \in L^2_{\mathbb{C}}(G), \qquad q \in G, \tag{5.7}$$

of G in a Hilbert space $L^2_{\mathbb{C}}(G)$ of square integrable complex functions on G. Note that the above mentioned coarsest topology on G is coarser then the original one, i.e., the representation (5.7) is strongly continuous. $\Box$

Let us consider unitary representations of a locally compact group G which are contained in its left-regular representation (5.7). In accordance with the expression (5.2), the corresponding representation $\Pi(h)$ of a group algebra $L^1_{\mathbb{C}}(G)$ in $L^2_{\mathbb{C}}(G)$ reads

$$(\Pi(h)f)(q) = \int h(g)(\Pi(g)f)(q)dg = \int h(g)f(g^{-1}q)dg = (h * f)(q). \tag{5.8}$$

Let G be a unimodular group. There is the following criterion that its unitary representation is contained in the left-regular one.

Theorem 5.4. If a continuous positive-definite function ψ on a unimodular locally compact group G is square integrable, then the representation π_ψ of G determined by ψ is contained in the left-regular representation Π (5.7) of G. Conversely, let π be a cyclic unitary representation of G which is contained in Π, and let θ be a cyclic vector for π. Then a continuous positive-definite function $\langle\pi(g)\theta|\theta\rangle$ on G is square integrable. $\Box$

The representation π_ψ in Theorem 5.4 is constructed as follows. Given a square integrable continuous positive-definite function ψ on G, there exists a positive-definite function $\theta \in L^2_{\mathbb{C}}(G)$ such that

$$\psi = \theta * \theta = \theta * \theta^* = \theta^* * \overline{\theta}^*.$$

This is a cyclic vector for π_ψ. The coefficients (5.1) of a representation π_ψ read

$$\varphi_{\xi,\eta}(g) = \overline{(\eta * \xi^*)(g)}.$$

In particular, if representations π_ψ and $\pi_{\psi'}$, determined by square integrable continuous positive-definite functions ψ and ψ' on G, are irreducible and inequivalent, then the corresponding cyclic vectors θ_ψ and $\theta_{\psi'}$ are orthogonal in $L^2_{\mathbb{C}}(G)$, while the functions ψ and ψ' fulfil the relations

$$\int \psi'(g)\overline{\psi}(g)dg = 0, \qquad \psi * \psi' = 0. \tag{5.9}$$

Now, let G be a connected locally compact (i.e., finite-dimensional) real Lie group. Any unitary representation of G yields a representation of its right Lie algebra $\mathfrak{g}$ as follows. In particular, a finite-dimensional unitary representation of G in a Hilbert space E is analytic, and a Lie algebra $\mathfrak{g}$ is represented by bounded operators in E.

If π is an infinite-dimensional (strongly continuous) unitary representation of G in a Hilbert space E, a representation of a Lie algebra $\mathfrak{g}$ fails to be defined everywhere on E in general. To construct a carrier space of $\mathfrak{g}$, let us consider a space $\mathcal{K}^\infty(G,\mathbb{C}) \subset L^1_{\mathbb{C}}(G)$ of smooth complex functions on G of compact support and the vectors

$$e_f = \pi^L(f)e = \int \pi(g)f(g)edg, \qquad e\in E, \qquad f\in \mathcal{K}^\infty(G,\mathbb{C}), \tag{5.10}$$

where π^L is the representation (5.3) of a group algebra $L^1_{\mathbb{C}}(G)$ [28]. The vectors e_f (5.10) exemplify smooth vectors of the representation π because, for any $\eta\in E$, the coefficients $\varphi_{e_f,\eta}(g)$ of π are smooth functions on G. The vectors e_f (5.10) for all $e\in E$ and $f\in\mathcal{K}^\infty(G,\mathbb{C})$ constitute a dense vector subspace E_∞ of E. Let u_a be a right-invariant vector field on G corresponding to an element $a\in\mathfrak{g}$. Then the assignment

$$\pi_\infty(a) : e_f \to \pi^L(u_a\rfloor df)e$$

provides a representation of a Lie algebra $\mathfrak{g}$ in E_∞.

6 GNS construction. Unbounded operators

There are quantum algebras (e.g., of quantum fields) whose representations in Hilbert spaces need not be normed. Therefore, generalizations of the conventional GNS representation of C^*-algebras (Theorem 2.7) to some classes of unnormed topological *-algebras has been studied [24, 27, 57].

In a general setting, by an operator in a Hilbert (or Banach) space E is meant a linear morphism a of a dense subspace $D(a)$ of E to E. The $D(a)$ is called the domain of an operator a. One says that an operator b on $D(b)$ is an extension of an operator a on $D(a)$ if $D(a)\subset D(b)$ and $b|_{D(a)} = a$. For the sake of brevity, we will write $a\subset b$. An operator a is said to be bounded on $D(a)$ if there exists a real number r such that

$$\|ae\|\leq r\|e\|, \qquad e\in D(a).$$

If otherwise, it is called unbounded. Any bounded operator on a domain $D(a)$ is uniquely extended to a bounded operator everywhere on E.

An operator a on a domain $D(a)$ is called closed if the condition that a sequence $\{e_i\}\subset D(a)$ converges to $e\in E$ and that the sequence $\{ae_i\}$ does to $e'\in E$ implies that $e\in D(a)$ and $e'=ae$. Of course, any operator defined everywhere on E is closed. An operator a on a domain $D(a)$ is called closable if it can be extended to a closed operator. The closure of a closable operator a is defined as the minimal closed extension of a.

Operators a and b in E are called adjoint if

$$\langle ae|e'\rangle = \langle e|be'\rangle, \quad e \in D(a), \quad e' \in D(b).$$

Any operator a has a maximal adjoint operator a^*, which is closed. Of course, $a \subset a^{**}$ and $b^* \subset a^*$ if $a \subset b$. An operator a is called symmetric if it is adjoint to itself, i.e., $a \subset a^*$. Hence, a symmetric operator is closable. One can obtain the following chain of extensions of a symmetric operator:

$$a \subset \overline{a} \subset a^{**} \subset a^* = \overline{a}^* = a^{***}.$$

In particular, if a is a symmetric operator, so are $\overline{a}$ and a^{**}. At the same time, the maximal adjoint operator a^* of a symmetric operator a need not be symmetric. A symmetric operator a is called self-adjoint if $a = a^*$, and it is called essentially self-adjoint if $\overline{a} = a^* = \overline{a}^*$. It should be emphasized that a symmetric operator a is sometimes called essentially self-adjoint if $a^{**} = a^*$. We here follow the terminology of [42]. If a is a closed operator, the both notions coincide. For bounded operators, the notions of symmetric, self-adjoint and essentially self-adjoint operators coincide.

Let E be a Hilbert space. A pair (B, D) of a dense subspace D of E and a unital algebra B of (unbounded) operators in E is called the Op^*-algebra (O^*-algebra in the terminology of [57]) on a domain D if, whenever $b \in B$, we have [27, 42]: (i) $D(b) = D$ and $bD \subset D$, (ii) $D \subset D(b^*)$, (iii) $b^*|_D \subset B$. An algebra B is provided with the involution $b \to b^+ = b^*|_D$, and its elements are closable.

A representation $\pi(A)$ of a *-algebra A in a Hilbert space E is defined as a homomorphism of A to an Op^*-algebra $(B, D(\pi))$ of (unbounded) operators in E such that $D(\pi) = D(\pi(a))$ for all $a \in A$ and this representation is Hermitian, i.e., $\pi(a^*) \subset \pi(a)^*$ for all $a \in A$. In this case, one also considers the representations

$$\overline{\pi} : a \to \overline{\pi}(a) = \overline{\pi(a)}|_{D(\overline{\pi})}, \qquad D(\overline{\pi}) = \bigcap_{a \in A} D(\overline{\pi(a)}),$$

$$\pi^* : a \to \pi^*(a) = \pi(a^*)^*|_{D(\pi^*)}, \qquad D(\pi^*) = \bigcap_{a \in A} D(\pi(a)^*),$$

called the closure of a representation π, and the adjoint representation, respectively. There are the representation extensions $\pi \subset \overline{\pi} \subset \pi^*$, where $\pi_1 \subset \pi_2$ means $D(\pi_1) \subset D(\pi_2)$. The representations $\overline{\pi}$ and π^{**} are Hermitian, while $\pi^* = \overline{\pi}^*$. A Hermitian representation $\pi(A)$ is said to be closed if $\pi = \overline{\pi}$, and it is self-adjoint if $\pi = \pi^*$. Herewith, a representation $\pi(A)$ is closed (resp. self-adjoint) if one of operators $\pi(A)$ is closed (resp. self-adjoint).

A representation domain $D(\pi)$ is endowed with the graph-topology. It is generated by neighborhoods of the origin

$$U(M, \varepsilon) = \{x \in D(\pi) \, : \, \sum_{a \in M} \|\pi(a)x\| < \varepsilon\},$$

where M is a finite subset of elements of A. All operators of $\pi(A)$ are continuous with respect to this topology. Let us note that the graph-topology is finer than the relative topology on $D(\pi) \subset E$, unless all operators $\pi(a)$, $a \in A$, are bounded [57].

Let $\overline{N}^g$ denote the closure of a subset $N \subset D(\pi)$ with respect to the graph-topology. An element $\theta \in D(\pi)$ is called strongly cyclic (cyclic in the terminology of [57]) if

$$D(\pi) \subset \overline{(\pi(A)\theta)}^g.$$

Then the GNS representation Theorem 2.7 can be generalized to Theorem 1.1 [27, 57].

Similarly to Remark 3.1, we say that representations π_1 and π_2 of a *-algebra A are equivalent if there exists an isomorphism γ of their carrier spaces such that

$$D(\pi_1) = \gamma(D(\pi_2)), \qquad \pi_1(a) = \gamma \circ \pi_2(a) \circ \gamma^{-1}, \qquad a \in A.$$

In particular, if representations are equivalent, their kernels coincide with each other.

Accordingly, states f and f' of a unital topological *-algebra A in Theorem 1.1 are called equivalent if they define equivalent representations π_f and$\pi_{f'}$.

By analogy with Theorem 3.2, one can show the following.

Theorem 6.1. Positive continuous forms f and f' on a unital topological *-algebra A are equivalent if there exist elements $b, b' \in A$ such that

$$f'(a) = f(b^+ab), \qquad f(a) = f'(b'^+ab'), \qquad a \in A.$$

□

We point out the particular class of nuclear barreled *-algebras. Let A be a locally convex topological *-algebra whose topology is defined by a set of multiplicative seminorms p_ι which satisfy the condition

$$p_\iota(a^*a) = p_\iota(a)^2, \qquad a \in A.$$

It is called a b^*-algebra. A unital b^*-algebra as like as a C^*-algebra is regular and symmetric, i.e., any element $(\mathbf{1} + a^*a)$, $a \in A$, is invertible and, moreover, $(\mathbf{1} + a^*a)^{-1}$ is bounded [1, 29]. The b^*-algebras are related to C^*-algebras as follows.

Theorem 6.2. Any b^*-algebra is the Hausdorff projective limit of a family of C^*-algebras, and *vice versa* [29]. □

In particular, every C^*-algebra A is a barreled b^*-algebra, i.e., every absorbing balanced closed subset is a neighborhood of the origin of A.

Let us additionally assume that A is a nuclear algebra, i.e., a nuclear space (Section 14.2). Then we have the following variant of the GNS representation Theorem 1.1 [29]

Theorem 6.3. Let A be a unital nuclear barreled b^*-algebra and f a positive continuous form on A. There exists a unique (up to unitary equivalence) cyclic representation π_f of A in a Hilbert space E_f by operators on a common invariant domain $D \subset E_f$. This domain can be topologized to conform a rigged Hilbert space such that all the operators representing A are continuous on D. □

Example 6.1. The following is an example of nuclear barreled b^*-algebras which is very familiar from quantum field theory [48, 49, 51]. Let Q be a nuclear space (Section 14.2). Let us consider a direct limit

$$\widehat{\otimes} Q = \mathbb{C} \oplus Q \oplus Q \widehat{\otimes} Q \oplus \cdots \oplus Q^{\widehat{\otimes} n} \oplus \cdots \tag{6.1}$$

of vector spaces

$$\widehat{\otimes}^{\leq n} Q = \mathbb{C} \oplus Q \oplus Q \widehat{\otimes} Q \oplus \cdots \oplus Q^{\widehat{\otimes} n}, \tag{6.2}$$

where $\widehat{\otimes}$ is the topological tensor product with respect to Grothendieck's topology (which coincides with the ε-topology on the tensor product of nuclear spaces [41]). The space (6.1) is provided

with the inductive limit topology, the finest topology such that the morphisms $\widehat{\otimes}^{\leq n} Q \to \widehat{\otimes} Q$ are continuous and, moreover, are imbeddings [61]. A convex subset V of $\widehat{\otimes} Q$ is a neighborhood of the origin in this topology iff $V \cap \widehat{\otimes}^{\leq n} Q$ is so in $\widehat{\otimes}^{\leq n} Q$. Furthermore, one can show that $\widehat{\otimes} Q$ is a unital nuclear barreled LF-algebra with respect to a tensor product [4]. The LF-property implies that a linear form f on $\widehat{\otimes} Q$ is continuous iff the restriction of f to each $\widehat{\otimes}^{\leq n} Q$ is so [61]. If a continuous conjugation $*$ is defined on Q, the algebra $\widehat{\otimes} Q$ is involutive with respect to the operation

$$* (q_1 \otimes \cdots \otimes q_n) = q_n^* \otimes \cdots \otimes q_1^* \tag{6.3}$$

on $Q^{\otimes n}$ extended by continuity and linearity to $Q^{\widehat{\otimes} n}$. Moreover, $\widehat{\otimes} Q$ is a b^*-algebra as follows. Since Q is a nuclear space, there is a family $\|.\|_k$, $k \in \mathbb{N}_+$, of continuous norms on Q. Let Q_k denote the completion of Q with respect to the norm $\|.\|_k$. Then one can show that the tensor algebra $\otimes Q_k$ is a C^*-algebra and that $\widehat{\otimes} Q$ (6.1) is a projective limit of these C^*-algebras with respect to morphisms $\otimes Q_{k+1} \to \otimes Q_k$ [29]. Since $\widehat{\otimes} Q$ (6.1) is a nuclear barreled b^*-algebra, it obeys GNS representation Theorem 6.3. Herewith, let us note that, due to the LF-property, a positive continuous form f on $\widehat{\otimes} Q$ is defined by a family of its restrictions f_n to tensor products $\widehat{\otimes}^{\leq n} Q$. One also can restrict a form f and the corresponding representation π_f to a tensor algebra

$$A_Q = \otimes Q \subset \widehat{\otimes} Q \tag{6.4}$$

of Q. However, a representation $\pi_f(A_Q)$ of A_Q in a Hilbert space E_f need not be cyclic. $\Box$

In quantum field theory, one usually choose Q the Schwartz space of functions of rapid decrease (Sections 9 and 10).

7 Example. Commutative nuclear groups

Following Example 6.1 in a case of a real nuclear space Q, let us consider a commutative tensor algebra

$$B_Q = \mathbb{R} \oplus Q \oplus Q \vee Q \oplus \cdots \oplus \overset{n}{\vee} Q \oplus \cdots . \tag{7.1}$$

Provided with the direct sum topology, B_Q becomes a unital topological *-algebra. It coincides with the universal enveloping algebra of the Lie algebra T_Q of an additive Lie group $T(Q)$ of translations in Q. Therefore, one can obtain the states of an algebra B_Q by constructing cyclic strongly continuous unitary representations of a nuclear Abelian group $T(Q)$.

Remark 7.1. Let us note that, in contrast to that studied in Section 5, a nuclear group $T(Q)$ is not locally compact, unless Q is finite-dimensional. $\diamondsuit$

A cyclic strongly continuous unitary representation π of $T(Q)$ in a Hilbert space $(E, \langle.|.\rangle_E)$ with a normalized cyclic vector $\theta \in E$ yields a complex function

$$Z(q) = \langle \pi(T(q))\theta|\theta\rangle_E$$

on Q. This function is proved to be continuous and positive-definite, i.e., $Z(0) = 1$ and

$$\sum_{i,j} Z(q_i - q_j)\overline{\lambda}_i \lambda_j \geq 0$$

for any finite set $q_1, \ldots, q_m$ of elements of Q and arbitrary complex numbers $\lambda_1, \ldots, \lambda_m$.

In accordance with the well-known Bochner theorem for nuclear spaces (Theorem 14.4), any continuous positive-definite function $Z(q)$ on a nuclear space Q is the Fourier transform

$$Z(q) = \int \exp[i\langle q, u\rangle]\mu(u) \tag{7.2}$$

of a positive measure μ of total mass 1 on the dual Q' of Q. Then the above mentioned representation π of $T(Q)$ can be given by the operators

$$T_Z(q)\rho(u) = \exp[i\langle q, u\rangle]\rho(u) \tag{7.3}$$

in a Hilbert space $L^2_{\mathbb{C}}(Q', \mu)$ of equivalence classes of square μ-integrable complex functions $\rho(u)$ on Q'. A cyclic vector θ of this representation is the μ-equivalence class $\theta \approx_\mu 1$ of a constant function $\rho(u) = 1$. Then we have

$$Z(q) = \langle T_Z(q)\theta|\theta\rangle_\mu = \int \exp[i\langle q, u\rangle]\mu. \tag{7.4}$$

Conversely, every positive measure μ of total mass 1 on the dual Q' of Q defines the cyclic strongly continuous unitary representation (7.3) of a group $T(Q)$. By virtue of the above mentioned Bochner theorem, it follows that every continuous positive-definite function $Z(q)$ on Q characterizes a cyclic strongly continuous unitary representation (7.3) of a nuclear Abelian group $T(Q)$. We agree to call $Z(q)$ the generating function of this representation.

Remark 7.2. The representation (7.3) need not be topologically irreducible. For instance, let $\rho(u)$ be a function on Q' such that a set where it vanishes is not a μ-null subset of Q'. Then the closure of a set $T_Z(Q)\rho$ is a $T(Q)$-invariant closed subspace of $L^2_{\mathbb{C}}(Q', \mu)$. ♢

Different generating functions $Z(q)$ determine inequivalent representations T_Z (7.3) of $T(Q)$ in general. One can show the following [22].

Theorem 7.1. Distinct generating functions $Z(q)$ and $Z'(q)$ determine equivalent representations T_Z and $T_{Z'}$ (7.3) of $T(Q)$ in Hilbert spaces $L^2_{\mathbb{C}}(Q', \mu)$ and $L^2_{\mathbb{C}}(Q', \mu')$ iff they are the Fourier transform of equivalent measures on Q'. □

Indeed, let

$$\mu' = s^2\mu, \tag{7.5}$$

where a function $s(u)$ is strictly positive almost everywhere on Q', and $\mu(s^2) = 1$. Then the map

$$L^2_{\mathbb{C}}(Q', \mu') \ni \rho(u) \to s(u)\rho(u) \in L^2_{\mathbb{C}}(Q', \mu) \tag{7.6}$$

provides an isomorphism between the representations $T_{Z'}$ and T_Z.

Similarly to the case of finite-dimensional Lie groups (Section 4), any strongly continuous unitary representation (7.3) of a nuclear group $T(Q)$ implies a representation of its Lie algebra by operators

$$\phi(q)\rho(u) = \langle q, u\rangle\rho(u) \tag{7.7}$$

in the same Hilbert space $L^2_{\mathbb{C}}(Q', \mu)$. Their mean values read

$$\langle\phi(q)\rangle = \omega_\theta(\phi(q)) = \langle\phi(q)\rangle = \int \langle q, u\rangle\mu(u). \tag{7.8}$$

The representation (7.7) is extended to that of the universal enveloping algebra B_Q (7.1).

Remark 7.3. Let us consider representations of $T(Q)$ with generating functions $Z(q)$ such that $\mathbb{R} \ni t \to Z(tq)$ is an analytic function on $\mathbb{R}$ at $t = 0$ for all $q \in Q$. Then one can show that a function $\langle q|u\rangle$ on Q' is square μ-integrable for all $q \in Q$ and that, consequently, the operators $\phi(q)$ (7.7) are bounded everywhere in a Hilbert space $L^2_{\mathbb{C}}(Q', \mu)$. Moreover, the corresponding mean values of elements of B_Q can be computed by the formula

$$\langle \phi(q_1)\cdots\phi(q_n)\rangle = i^{-n}\frac{\partial}{\partial\alpha^1}\cdots\frac{\partial}{\partial\alpha^n}Z(\alpha^i q_i)|_{\alpha^i=0} = \int \langle q_1, u\rangle\cdots\langle q_n, u\rangle\mu(u). \tag{7.9}$$

$\diamondsuit$

8 Infinite canonical commutation relations

The canonical commutation relations (CCR) are of central importance in AQT as a method of canonical quantization. A remarkable result about CCR for finite degrees of freedom is the Stone–von Neumann uniqueness theorem which states that all irreducible representations of these CCR are unitarily equivalent [40]. On the contrary, CCR of infinite degrees of freedom admit infinitely many inequivalent irreducible representations [21].

One can provide the comprehensive description of representations of CCR modelled over an infinite-dimensional nuclear space Q [22, 24, 49].

Let Q be a real nuclear space endowed with a non-degenerate separately continuous Hermitian form $\langle .|.\rangle$. This Hermitian form brings Q into a Hausdorff pre-Hilbert space. A nuclear space Q, the completion $\widetilde{Q}$ of a pre-Hilbert space Q, and the dual Q' of Q make up the rigged Hilbert space $Q \subset \widetilde{Q} \subset Q'$ (14.8).

Let us consider a group $G(Q)$ of triples $g = (q_1, q_2, \lambda)$ of elements q_1, q_2 of Q and complex numbers λ, $|\lambda| = 1$, which are subject to multiplications

$$(q_1, q_2, \lambda)(q_1', q_2', \lambda') = (q_1 + q_1', q_2 + q_2', \exp[i\langle q_2, q_1'\rangle]\lambda\lambda'). \tag{8.1}$$

It is a Lie group whose group space is a nuclear manifold modelled over a vector space $W = Q \oplus Q \oplus \mathbb{R}$. Let us denote $T(q) = (q, 0, 0)$, $P(q) = (0, q, 0)$. Then the multiplication law (8.1) takes a form

$$T(q)T(q') = T(q + q'), \quad P(q)P(q') = P(q + q'), \quad P(q)T(q') = \exp[i\langle q|q'\rangle]T(q')P(q). \tag{8.2}$$

Written in this form, $G(Q)$ is called the nuclear Weyl CCR group.

The complexified Lie algebra of a nuclear Lie group $G(Q)$ is the unital Heisenberg CCR algebra $\mathfrak{g}(Q)$. It is generated by elements I, $\phi(q)$, $\pi(q)$, $q \in Q$, which obey the Heisenberg CCR

$$[\phi(q), I] = \pi(q), I] = [\phi(q), \phi(q')] = [\pi(q), \pi(q')] = 0, \qquad [\pi(q), \phi(q')] = -i\langle q|q'\rangle I. \tag{8.3}$$

There is the exponential map

$$T(q) = \exp[i\phi(q)], \qquad P(q) = \exp[i\pi(q)].$$

Due to the relation (14.7), the normed topology on a pre-Hilbert space Q defined by a Hermitian form $\langle .|.\rangle$ is coarser than the nuclear space topology. The latter is metric, separable

and, consequently, second-countable. Hence, a pre-Hilbert space Q also is second-countable and, therefore, admits a countable orthonormal basis. Given such a basis $\{q_i\}$ for Q, the Heisenberg CCR (8.3) take a form

$$[\phi(q_j), I] = \pi(q_j), I] = [\phi(q_j), \phi(q_k)] = [\pi(q_k), \pi(q_j)] = 0, \qquad [\pi(q_j), \phi(q_k)] = -i\delta_{jk}I.$$

A Weyl CCR group $G(Q)$ contains two nuclear Abelian subgroups $T(Q)$ and $P(Q)$. Following the representation algorithm in [22], we first construct representations of a nuclear Abelian group $T(Q)$. Then these representations under certain conditions can be extended to representations of a Weyl CCR group $G(Q)$.

Following Section 7, we treat a nuclear Abelian group $T(Q)$ as being a group of translations in a nuclear space Q. Let us consider its cyclic strongly continuous unitary representation T_Z (7.3) in a Hilbert space $L^2_{\mathbb{C}}(Q', \mu)$ of equivalence classes of square μ-integrable complex functions $\rho(u)$ on the dual Q' of Q which is defined by the generating function Z (7.2). This representation can be extended to a Weyl CCR group $G(Q)$ if a measure μ possesses the following property.

Let u_q, $q \in Q$, be an element of Q' given by the condition $\langle q', u_q\rangle = \langle q'|q\rangle$, $q' \in Q$. These elements form the range of a monomorphism $Q \to Q'$ determined by a Hermitian form $\langle .|.\rangle$ on Q. Let a measure μ in the expression (7.2) remain equivalent under translations

$$Q' \ni u \to u + u_q \in Q', \qquad u_q \in Q \subset Q',$$

in Q', i.e.,

$$\mu(u + u_q) = a^2(q, u)\mu(u), \qquad u_q \in Q \subset Q', \tag{8.4}$$

where a function $a(q, u)$ is square μ-integrable and strictly positive almost everywhere on Q'. This function fulfils the relations

$$a(0, u) = 1, \qquad a(q + q', u) = a(q, u)a(q', u + u_q). \tag{8.5}$$

A measure on Q' obeying the condition (8.4) is called translationally quasi-invariant, though it does not remain equivalent under an arbitrary translation in Q', unless Q is finite-dimensional.

Let the generating function Z (7.2) of a cyclic strongly continuous unitary representation of a nuclear group $T(Q)$ be the Fourier transform of a translationally quasi-invariant measure μ on Q'. Then one can extend the representation (7.3) of this group to a unitary strongly continuous representation of a Weyl CCR group $G(Q)$ in a Hilbert space $L^2_{\mathbb{C}}(Q', \mu)$ by the operators (5.6) in Example 5.2. These operators read

$$P_Z(q)\rho(u) = a(q, u)\rho(u + u_q). \tag{8.6}$$

Herewith, the following is true.

Theorem 8.1. Equivalent representations of a group $T(Q)$ are extended to equivalent representations of a Weyl CCR group $G(Q)$. □

Proof. Let μ' (7.5) be a μ-equivalent positive measure of total mass 1 on Q'. The equality

$$\mu'(u + u_q) = s^{-2}(u)a^2(q, u)s^2(u + u_q)\mu'(u)$$

shows that it also is translationally quasi-invariant. Then the isomorphism (7.6) between representations T_Z and $T_{Z'}$ of a nuclear Abelian group $T(Q)$ is extended to the isomorphism

$$P_{Z'}(q) = s^{-1}P_Z(q)s : \rho(u) \to s^{-1}(u)a(q, u)s(u + u_q)\rho(u + u_q)$$

of the corresponding representations of a Weyl CCR group $G(Q)$. $\diamondsuit$

Similarly to the case of finite-dimensional Lie groups (Section 4), any strongly continuous unitary representation T_Z (7.3), P_Z (8.6) of a nuclear Weyl CCR group $G(Q)$ implies a representation of its Lie algebra $\mathfrak{g}(Q)$ by (unbounded) operators in the same Hilbert space $L^2_{\mathbb{C}}(Q',\mu)$ [24, 49]. This representation reads

$$\phi(q)\rho(u) = \langle q,u\rangle\rho(u), \qquad \pi(q)\rho(u) = -i(\delta_q + \eta(q,u))\rho(u), \tag{8.7}$$

$$\delta_q\rho(u) = \lim_{\alpha\to 0}\alpha^{-1}[\rho(u+\alpha u_q) - \rho(u)], \qquad \alpha\in\mathbb{R},$$

$$\eta(q,u) = \lim_{\alpha\to 0}\alpha^{-1}[a(\alpha q,u) - 1]. \tag{8.8}$$

It follows at once from the relations (8.5) that

$$\delta_q\delta_{q'} = \delta_{q'}\delta_q, \qquad \delta_q(\eta(q',u)) = \delta_{q'}(\eta(q,u)),$$
$$\delta_q = -\delta_{-q}, \qquad \delta_q(\langle q',u\rangle) = \langle q'|q\rangle,$$
$$\eta(0,u) = 0, \quad u\in Q', \qquad \delta_q\theta = 0, \quad q\in Q.$$

Then it is easily justified that the operators (8.7) fulfil the Heisenberg CCR (8.3). The unitarity condition implies the conjugation rule

$$\langle q,u\rangle^* = \langle q,u\rangle, \qquad \delta_q^* = -\delta_q - 2\eta(q,u).$$

Hence, the operators (8.7) are Hermitian.

The operators $\pi(q)$ (8.7), unlike $\phi(q)$, act in a subspace E_∞ of all smooth complex functions in $L^2_{\mathbb{C}}(Q',\mu)$ whose derivatives of any order also belongs to $L^2_{\mathbb{C}}(Q',\mu)$. However, E_∞ need not be dense in a Hilbert space $L^2_{\mathbb{C}}(Q',\mu)$, unless Q is finite-dimensional.

A space E_∞ also is a carrier space of a representation of the universal enveloping algebra $\overline{\mathfrak{g}}(Q)$ of a Heisenberg CCR algebra $\mathfrak{g}(Q)$. The representations of $\mathfrak{g}(Q)$ and $\overline{\mathfrak{g}}(Q)$ in E_∞ need not be irreducible. Therefore, let us consider a subspace $E_\theta = \overline{\mathfrak{g}}(Q)\theta$ of E_∞, where θ is a cyclic vector for a representation of a Weyl CCR group in $L^2_{\mathbb{C}}(Q',\mu)$. Obviously, a representation of a Heisenberg CCR algebra $\mathfrak{g}(Q)$ in E_θ is algebraically irreducible.

One also introduces creation and annihilation operators

$$a^\pm(q) = \frac{1}{\sqrt{2}}[\phi(q)\mp i\pi(q)] = \frac{1}{\sqrt{2}}[\mp\delta_q \mp \eta(q,u) + \langle q,u\rangle]. \tag{8.9}$$

They obey the conjugation rule $(a^\pm(q))^* = a^\mp(q)$ and the commutation relations

$$[a^-(q), a^+(q')] = \langle q|q'\rangle\mathbf{1}, \qquad [a^+(q),a^+(q')] = [a^-(q),a^-(q')] = 0.$$

The particle number operator N in a carrier space E_θ is defined by conditions

$$[N, a^\pm(q)] = \pm a^\pm(q)$$

up to a summand $\lambda\mathbf{1}$. With respect to a countable orthonormal basis $\{q_k\}$, this operator N is given by a sum

$$N = \sum_k a^+(q_k)a^-(q_k), \tag{8.10}$$

but need not be defined everywhere in E_θ, unless Q is finite-dimensional.

Gaussian measures given by the Fourier transform (14.28) exemplify a physically relevant class of translationally quasi-invariant measures on the dual Q' of a nuclear space Q. Their Fourier transforms obey the analiticity condition in Remark 7.3.

Let μ_K denote a Gaussian measure on Q' whose Fourier transform is a generating function

$$Z_K = \exp[-\frac{1}{2}M_K(q)] \tag{8.11}$$

with the covariance form

$$M_K(q) = \langle K^{-1}q|K^{-1}q\rangle, \tag{8.12}$$

where K is a bounded invertible operator in the Hilbert completion $\widetilde{Q}$ of Q with respect to a Hermitian form $\langle .|.\rangle$. The Gaussian measure μ_K is translationally quasi-invariant, i.e.,

$$\mu_K(u+u_q) = a_K^2(q,u)\mu_K(u).$$

Using the formula (7.9), one can show that

$$a_K(q,u) = \exp[-\frac{1}{4}M_K(Sq) - \frac{1}{2}\langle Sq,u\rangle], \tag{8.13}$$

where $S = KK^*$ is a bounded Hermitian operator in $\widetilde{Q}$.

Let us construct a representation of a CCR algebra $\mathfrak{g}(Q)$ determined by the generating function Z_K (8.11). Substituting the function (8.13) into the formula (8.8), we obtain

$$\eta(q,u) = -\frac{1}{2}\langle Sq,u\rangle.$$

Hence, the operators $\phi(q)$ and $\pi(q)$ (8.7) take a form

$$\phi(q) = \langle q,u\rangle, \qquad \pi(q) = -i(\delta_q - \frac{1}{2}\langle Sq,u\rangle). \tag{8.14}$$

Accordingly, the creation and annihilation operators (8.9) read

$$a^{\pm}(q) = \frac{1}{\sqrt{2}}[\mp\delta_q \pm \frac{1}{2}\langle Sq,u\rangle + \langle q,u\rangle]. \tag{8.15}$$

They act on the subspace E_θ, $\theta \approx_{\mu_K} 1$, of a Hilbert space $L^2_{\mathbb{C}}(Q',\mu_K)$, and they are Hermitian with respect to a Hermitian form $\langle .|.\rangle_{\mu_K}$ on $L^2_{\mathbb{C}}(Q',\mu_K)$.

Remark 8.1. If a representation of CCR is characterized by the Gaussian generating function (8.11), it is convenient for a computation to express all operators into the operators δ_q and $\phi(q)$, which obey commutation relations

$$[\delta_q, \phi(q')] = \langle q'|q\rangle.$$

For instance, we have

$$\pi(q) = -i\delta_q - \frac{i}{2}\phi(Sq).$$

The mean values $\langle\phi(q_1)\cdots\phi(q_n)\delta_q\rangle$ vanish, while the meanvalues $\langle\phi(q_1)\cdots\phi(q_n)\rangle$, defined by the formula (7.9), obey the Wick theorem relations

$$\langle\phi(q_1)\cdots\phi(q_n)\rangle = \sum\langle\phi(q_{i_1})\phi(q_{i_2})\rangle\cdots\langle\phi(q_{i_{n-1}})\phi(q_{i_n})\rangle, \tag{8.16}$$

where the sum runs through all partitions of a set $1, \ldots, n$ in ordered pairs $(i_1 < i_2), \ldots (i_{n-1} < i_n)$, and where

$$\langle \phi(q)\phi(q')\rangle = \langle K^{-1}q|K^{-1}q'\rangle.$$

$\diamondsuit$

In particular, let us put $K = \sqrt{2}\cdot\mathbf{1}$. Then the generating function (8.11) takes a form

$$Z_{\mathrm{F}}(q) = \exp[-\frac{1}{4}\langle q|q\rangle], \tag{8.17}$$

and defines the Fock representation of a Heisenberg CCR algebra $\mathfrak{g}(Q)$:

$$\begin{aligned} &\phi(q) = \langle q, u\rangle, \qquad \pi(q) = -i(\delta_q - \langle q, u\rangle), \\ &a^+(q) = \frac{1}{\sqrt{2}}[-\delta_q + 2\langle q, u\rangle], \qquad a^-(q) = \frac{1}{\sqrt{2}}\delta_q. \end{aligned} \tag{8.18}$$

Its carrier space is the subspace E_θ, $\theta \approx_{\mu_{\mathrm{F}}} 1$ of the Hilbert space $L^2_{\mathbb{C}}(Q', \mu_{\mathrm{F}})$, where μ_{F} denotes a Gaussian measure whose Fourier transform is (8.17). We agree to call it the Fock measure.

The Fock representation (8.18) up to an equivalence is characterized by the existence of a cyclic vector θ such that

$$a^-(q)\theta = 0, \qquad q \in Q. \tag{8.19}$$

For the representation in question, this is $\theta \approx_{\mu_{\mathrm{F}}} 1$. An equivalent condition is that the particle number operator N (8.10) exists and its spectrum is lower bounded. The corresponding eigenvector of N in E_θ is θ itself so that $N\theta = 0$. Therefore, it is treated a particleless vacuum.

A glance at the expression (8.15) shows that the condition (8.19) does not hold, unless Z_K is Z_{F} (8.17). For instance, the particle number operator in the representation (8.15) reads

$$\begin{aligned} N = \sum_j a^+(q_j)a^-(q_j) = \sum_j [-\delta_{q_j}\delta_{q_j} + S^j_k\langle q_k, u\rangle\partial_{q_j} + \\ (\delta_{km} - \frac{1}{4}S^j_k S^j_m)\langle q_k, u\rangle\langle q_m, u\rangle - (\delta_{jj} - \frac{1}{2}S^j_j)], \end{aligned}$$

where $\{q_k\}$ is an orthonormal basis for a pre-Hilbert space Q. One can show that this operator is defined everywhere on E_θ and is lower bounded only if the operator S is a sum of the scalar operator $2\cdot\mathbf{1}$ and a nuclear operator in $\widetilde{Q}$, in particular, if

$$\mathrm{Tr}(\mathbf{1} - \frac{1}{2}S) < \infty.$$

This condition also is sufficient for measures μ_K and μ_F (and, consequently, the corresponding representations) to be equivalent [22]. For instance, a generating function

$$Z_c(q) = \exp[-\frac{c^2}{2}\langle q|q\rangle], \qquad c^2 \neq \frac{1}{2},$$

defines a non-Fock representation of nuclear CCR.

Remark 8.2. Since the Fock measure μ_{F} on Q' remains equivalent only under translations by vectors $u_q \in Q \subset Q'$, the measure

$$\mu_\sigma(u) = \mu_{\mathrm{F}}(u + \sigma), \qquad \sigma \in Q' \setminus Q,$$

on Q' determines a non-Fock representation of nuclear CCR. Indeed, this measure is translationally quasi-invariant:

$$\mu_\sigma(u+u_q) = a^2_\sigma(q,u)\mu_\sigma(u), \qquad a_\sigma(q,u) = a_{\rm F}(q,u-\sigma),$$

and its Fourier transform

$$Z_\sigma(q) = \exp[i\langle q,\sigma\rangle]Z_{\rm F}(q)$$

is a positive-definite continuous function on Q. Then the corresponding representation of a CCR algebra is given by operators

$$a^+(q) = \frac{1}{\sqrt{2}}(-\delta_q + 2\langle q,u\rangle - \langle q,\sigma\rangle), \qquad a^-(q) = \frac{1}{\sqrt{2}}(\delta_q + \langle q,\sigma\rangle). \tag{8.20}$$

In comparison with the all above mentioned representations, these operators possess non-vanishing vacuum mean values

$$\langle a^\pm(q)\theta|\theta\rangle_{\mu_{\rm F}} = \langle q,\sigma\rangle.$$

If $\sigma \in Q \subset Q'$, the representation (8.20) becomes equivalent to the Fock representation (8.18) due to a morphism

$$\rho(u) \to \exp[-\langle\sigma,u\rangle]\rho(u+\sigma).$$

$\diamondsuit$

Remark 8.3. Let us note that the non-Fock representation (8.14) of the CCR algebra (8.3) in a Hilbert space $L^2_{\mathbb{C}}(Q',\mu_K)$ is the Fock representation

$$\phi_K(q) = \phi(q) = \langle q,u\rangle, \qquad \pi_K(q) = \pi(S^{-1}q) = -i(\delta^K_q - \frac{1}{2}\langle q,u\rangle), \qquad \delta^K_q = \delta_{S^{-1}q},$$

of a CCR algebra $\{\phi_K(q),\pi_K(q),I\}$, where

$$[\phi_K(q),\pi_K(q)] = i\langle K^{-1}q|K^{-1}q'\rangle I.$$

$\diamondsuit$

9 Free quantum fields

There are two main algebraic formulation of QFT. In the framework of the first one, called local QFT, one associates to a certain class of subsets of a Minkowski space a net of von Neumann, C^*- or Op^*-algebras which obey certain axioms [2, 15, 25, 26, 27]. Its inductive limit is called either a global algebra (in the case of von Neumann algebras) or a quasilocal algebra (for a net of C^*-algebras). This construction is extended to non-Minkowski spaces, e.g., globally hyperbolic spacetimes [13, 14, 45].

In a different formulation of algebraic QFT with reference to the field-particle dualism, realistic quantum field models are described by tensor algebras, as a rule.

Let Q be a nuclear space. Let us consider the direct limit $\widehat\otimes Q$ (6.1) of the vector spaces $\widehat\otimes^{\leq n} Q$ (6.2) where $\widehat\otimes$ is the topological tensor product with respect to Grothendieck's topology. As was

mentioned above, provided with the inductive limit topology, the tensor algebra $\widehat{\otimes}Q$ (6.1) is a unital nuclear b^*-algebra (Example 6.1). Therefore, one can apply GNS representation Theorem 6.3 to it. A state f of this algebra is given by a tuple $\{f_n\}$ of continuous forms on the tensor algebra A_Q (6.4). Its value $f(q^1\cdots q^n)$ are interpreted as the vacuum expectation of a system of fields $q^1,\ldots,q^n$.

In algebraic QFT, one usually choose by Q the Schwartz space of functions of rapid decrease.

Remark 9.1. By functions of rapid decrease on an Euclidean space $\mathbb{R}^n$ are called complex smooth functions $\psi(x)$ such that the quantities

$$\|\psi\|_{k,m}=\max_{|\alpha|\leq k}\sup_x (1+x^2)^m|D^\alpha\psi(x)| \tag{9.1}$$

are finite for all $k,m\in\mathbb{N}$. Here, we follow the standard notation

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x^1\cdots\partial^{\alpha_n}x^n}, \qquad |\alpha|=\alpha_1+\cdots+\alpha_n,$$

for an n-tuple of natural numbers $\alpha=(\alpha_1,\ldots,\alpha_n)$. The functions of rapid decrease constitute a nuclear space $S(\mathbb{R}^n)$ with respect to the topology determined by the seminorms (9.1). Its dual is a space $S'(\mathbb{R}^n)$ of tempered distributions [7, 22, 41]. The corresponding contraction form is written as

$$\langle\psi,h\rangle=\int\psi(x)h(x)d^nx, \qquad \psi\in S(\mathbb{R}^n), \qquad h\in S'(\mathbb{R}^n).$$

A space $S(\mathbb{R}^n)$ is provided with a non-degenerate separately continuous Hermitian form

$$\langle\psi|\psi'\rangle=\int\psi(x)\overline{\psi'(x)}d^nx.$$

The completion of $S(\mathbb{R}^n)$ with respect to this form is a space $L^2_C(\mathbb{R}^n)$ of square integrable complex functions on $\mathbb{R}^n$. We have a rigged Hilbert space

$$S(\mathbb{R}^n)\subset L^2_C(\mathbb{R}^n)\subset S'(\mathbb{R}^n).$$

Let $\mathbb{R}_n$ denote the dual of $\mathbb{R}^n$ coordinated by (p_λ). The Fourier transform

$$\psi^F(p)=\int\psi(x)e^{ipx}d^nx, \qquad px=p_\lambda x^\lambda, \tag{9.2}$$

$$\psi(x)=\int\psi^F(p)e^{-ipx}d_np, \qquad d_np=(2\pi)^{-n}d^np, \tag{9.3}$$

defines an isomorphism between the spaces $S(\mathbb{R}^n)$ and $S(\mathbb{R}_n)$. The Fourier transform of tempered distributions is given by the condition

$$\int h(x)\psi(x)d^nx=\int h^F(p)\psi^F(-p)d_np,$$

and it is written in the form (9.2) – (9.3). It provides an isomorphism between the spaces of tempered distributions $S'(\mathbb{R}^n)$ and $S'(\mathbb{R}_n)$. $\diamondsuit$

For the sake of simplicity, we here restrict our consideration to real scalar fields and choose by Q the real subspace RS^4 of the Schwartz space $S(\mathbb{R}^4)$ of smooth complex functions of rapid decrease on $\mathbb{R}^4$ [49]. Since a subset $\overset{n}{\otimes} S(\mathbb{R}^k)$ is dense in $S(\mathbb{R}^{kn})$, we henceforth identify the tensor algebra A_{RS^4} (6.4) of a nuclear space RS^4 with the algebra

$$A=\mathbb{R}\oplus RS^4\oplus RS^8\oplus\cdots, \tag{9.4}$$

called the Borchers algebra [8, 27, 49]. Any state f of this algebra is represented by a collection of tempered distributions $\{W_k \in S'(\mathbb{R}^{4k})\}$ by the formula

$$f(\psi_k) = \int W_k(x_1, \ldots, x_k)\psi_k(x_1, \ldots, x_k)d^4x_1 \cdots d^4x_k, \qquad \psi_k \in RS^{4k}.$$

For instance, the states of scalar quantum fields in a Minkowski space $\mathbb{R}^4$ are described by the Wightman functions $W_n \subset S'(\mathbb{R}^{4k})$ in the Minkowski space which obey the Garding–Wightman axioms of axiomatic QFT [7, 64, 66]. Let us mention the Poincaré covariance axiom, the condition of the existence and uniqueness of a vacuum θ_0, and the spectrum condition. They imply that: (i) a carrier Hilbert space E_W of Wightman quantum fields admits a unitary representation of a Poinaré group, (ii) a space E_W contains a unique (up to scalar multiplications) vector ψ_0, called the vacuum vector, invariant under Poincaré transformations, (iii) the spectrum of an energy-momentum operator lies in the closed positive light cone. In particular, the Poincaré covariance condition implies the translation invariance and the Lorentz covariance of Wightman functions. Due to the translation invariance of Wightman functions W_k, there exist tempered distributions $w_k \in S'(\mathbb{R}^{4k-4})$, also called Wightman functions, such that

$$W_k(x_1, \ldots, x_k) = w_k(x_1 - x_2, \ldots, x_{k-1} - x_k). \tag{9.5}$$

Note that Lorentz covariant tempered distributions for one argument only are well described [7, 67]. In order to modify Wightman's theory, one studies different classes of distributions which Wightman functions belong to [59, 60].

Let us here focus on states of the Borchers algebra A (9.4) which describe free quantum scalar fields of mass m [49, 51].

Let us provide a nuclear space RS^4 with a positive complex bilinear form

$$(\psi|\psi')_m = \frac{2}{i}\int \psi(x)D_m^-(x-y)\psi'(y)d^4xd^4y = \int \psi^F(-\omega, -\vec{p})\psi'^F(\omega, \vec{p})\frac{d_3p}{\omega}, \tag{9.6}$$

$$D_m^-(x) = i(2\pi)^{-3}\int \exp[-ipx]\theta(p_0)\delta(p^2 - m^2)d^4p, \tag{9.7}$$

$$\omega = (\vec{p}^{\,2} + m^2)^{1/2},$$

where p^2 is the Minkowski square, $\theta(p_0)$ is the Heaviside function, and $D_m^-(x)$ is the negative frequency part of the Pauli–Jordan function

$$D_m(x) = i(2\pi)^{-3}\int \exp[-ipx](\theta(p_0) - \theta(-p_0))\delta(p^2 - m^2)d^4p. \tag{9.8}$$

Since a function $\psi(x)$ is real, its Fourier transform (9.2) satisfies an equality $\psi^F(p) = \overline{\psi}^F(-p)$.

The bilinear form (9.6) is degenerate because the Pauli–Jordan function $D_m^-(x)$ obeys a mass shell equation

$$(\Box + m^2)D_m^-(x) = 0.$$

It takes non-zero values only at elements $\psi^F \in RS_4$ which are not zero on a mass shell $p^2 = m^2$. Therefore, let us consider the quotient space

$$\gamma_m : RS^4 \to RS^4/J, \tag{9.9}$$

where

$$J = \{\psi \in RS^4 \,:\, (\psi|\psi)_m = 0\}$$

is the kernel of the square form (9.6). The map γ_m (9.9) assigns the couple of functions $(\psi^F(\omega,\vec{p}),\psi^F(-\omega,\vec{p}))$ to each element $\psi\in RS^4$ with a Fourier transform $\psi^F(p_0,\vec{p})\in RS_4$. Let us equip the factor space RS^4/J with a real bilinear form

$$(\gamma\psi|\gamma\psi')_L = \mathrm{Re}(\psi|\psi') = \tag{9.10}$$
$$\frac{1}{2}\int[\psi^F(-\omega,-\vec{p})\psi'^F(\omega,\vec{p})+\psi^F(\omega,-\vec{p})\psi'^F(-\omega,\vec{p})]\frac{d_3\vec{p}}{\omega}.$$

Then it is decomposed into a direct sum $RS^4/J=L^+\oplus L^-$ of subspaces

$$L^\pm=\{\psi^F_\pm(\omega,\vec{p})=\frac{1}{2}(\psi^F(\omega,\vec{p})\pm\psi^F(-\omega,\vec{p}))\},$$

which are mutually orthogonal with respect to the bilinear form (9.10).

There exist continuous isometric morphisms

$$\gamma_+:\psi^F_+(\omega,\vec{p})\to q^F(\vec{p})=\omega^{-1/2}\psi^F_+(\omega,\vec{p}),\qquad \gamma_-:\psi^F_-(\omega,\vec{p})\to q^F(\vec{p})=-i\omega^{-1/2}\psi^F_-(\omega,\vec{p})$$

of spaces L^+ and L^- to a nuclear space RS^3 endowed with a non-degenerate separately continuous Hermitian form

$$\langle q|q'\rangle=\int q^F(-\vec{p})q'^F(\vec{p})d_3p. \tag{9.11}$$

It should be emphasized that the images $\gamma_+(L^+)$ and $\gamma_-(L^-)$ in RS^3 are not orthogonal with respect to the scalar form (9.11). Combining γ_m (9.9) and $\gamma_\pm$, we obtain continuous morphisms $\tau_\pm:RS^4\to RS^3$ given by the expressions

$$\tau_+(\psi)=\gamma_+(\gamma_m\psi)_+=\frac{1}{2\omega^{1/2}}\int[\psi^F(\omega,\vec{p})+\psi^F(-\omega,\vec{p})]\exp[-i\vec{p}\,\vec{x}]d_3p,$$
$$\tau_-(\psi)=\gamma_-(\gamma_m\psi)_-=\frac{1}{2i\omega^{1/2}}\int[\psi^F(\omega,\vec{p})-\psi^F(-\omega,\vec{p})]\exp[-i\vec{p}\,\vec{x}]d_3p.$$

Now let us consider a Heisenberg CCR algebra

$$\mathfrak{g}(RS^3)=\{(\phi(q),\pi(q)),I,\,q\in RS^3\} \tag{9.12}$$

modelled over a nuclear space RS^3, which is equipped with the Hermitian form (9.11) (Section 8). Using the morphisms $\tau_\pm$, let us define a map

$$\Gamma_m:RS^4\ni\psi\to\phi(\tau_+(\psi))-\pi(\tau_-(\psi))\in\mathfrak{g}(RS^3). \tag{9.13}$$

With this map, one can think of (9.12) as being the algebra of instantaneous CCR of scalar fields on a Minkowski space $\mathbb{R}^4$. Owing to the map (9.13), any representation of the Heisenberg CCR algebra $\mathfrak{g}(RS^3)$ (9.12) defined by a translationally quasi-invariant measure μ on $S'(\mathbb{R}^3)$ induces a state

$$f_m(\psi^1\cdots\psi^n)=\langle\phi(\tau_+(\psi^1))+\pi(\tau_-(\psi^1))]\cdots[\phi(\tau_+(\psi^n))+\pi(\tau_-(\psi^n))]\rangle \tag{9.14}$$

of the Borchers algebra A (9.4). Furthermore, one can justify that the corresponding distributions W_n fulfil the mass shell equation and that the following commutation relation holds:

$$W_2(x,y)-W_2(y,x)=-iD_m(x-y),$$

where $D_m(x-y)$ is the Pauli–Jordan function (9.8). Thus, the state (9.14) of the Borchers algebra A (9.4) describes quantum scalar fields of mass m.

For instance, let us consider the Fock representation $Z_{\rm F}(q)$ (8.17) of the Heisenberg CCR algebra $\mathfrak{g}(RS^3)$ (9.12). Using the formulae in Remark 8.1 where a form $\langle q|q'\rangle$ is given by the expression (9.11), one observes that the state f_m (9.14) satisfies the Wick theorem relations

$$f_m(\psi^1\cdots\psi^n) = \sum_{(i_1\ldots i_n)} f_2(\psi^{i_1}\psi^{i_2})\cdots f_2(\psi^{i_{n-1}}\psi^{i_n}), \tag{9.15}$$

where a state f_2 is given by the Wightman function

$$W_2(x,y) = \frac{1}{i}D^-_m(x-y). \tag{9.16}$$

Thus, the state f_m (9.15) describes free quantum scalar fields of mass m.

Similarly, one can obtain states of the Borchers algebra A (9.4) generated by non-Fock representations (8.11) of the instantaneous CCR algebra $\mathfrak{g}(RS^3)$, e.g., if $K^{-1} = c\mathbf{1} \neq 2^{-1/2}\mathbf{1}$. These states fail to be defined by Wightman functions.

It should be emphasized that, given a different mass m', we have a different map $\Gamma_{m'}$ (9.13) of the Borchers algebra A (9.4) to the Heisenberg CCR algebra $\mathfrak{g}(RS^3)$ (9.12). Accordingly, the Fock representation $Z_{\rm F}(q)$ (8.17) of the Heisenberg CCR algebra $\mathfrak{g}(RS^3)$ (9.12) yields the state $f_{m'}$ (9.15) where a state f_2 is given by the Wightman function

$$W_2(x,y) = \frac{1}{i}D^-_{m'}(x-y). \tag{9.17}$$

If $m \neq m'$, the states f_m and $f_{m'}$ (9.15) the Borchers algebra A (9.4) are inequivalent because its representations Γ_m and $\Gamma_{m'}$ (9.13) possess different kernels.

10 Euclidean QFT

In QFT, interacting quantum fields created at some instant and annihilated at another one are described by complete Green functions. They are given by the chronological functionals

$$f^c(\psi_k) = \int W^c_k(x_1,\ldots,x_k)\psi_k(x_1,\ldots,x_k)d^4x_1\cdots d^4x_k, \qquad \psi_k \in RS^{4k}, \tag{10.1}$$

$$W^c_k(x_1,\ldots,x_k) = \sum_{(i_1\ldots i_k)} \theta(x^0_{i_1}-x^0_{i_2})\cdots\theta(x^0_{i_{k-1}}-x^0_{i_n})W_k(x_1,\ldots,x_k), \tag{10.2}$$

where $W_k \in S'(\mathbb{R}^{4k})$ are tempered distributions, and the sum runs through all permutations $(i_1\ldots i_k)$ of the tuple of numbers $1,\ldots,k$ [6].

A problem is that the functionals W^c_k (10.2) need not be tempered distributions. For instance, $W^c_1 \in S'(\mathbb{R})$ iff $W_1 \in S'(\mathbb{R}_\infty)$, where $\mathbb{R}_\infty$ is the compactification of $\mathbb{R}$ by means of a point $\{+\infty\} = \{-\infty\}$ [7]. Moreover, chronological forms are not positive. Therefore, they do not provide states of the Borchers algebra A (9.4) in general.

At the same time, the chronological forms (10.2) come from the Wick rotation of Euclidean states of the Borchers algebra [48, 49, 51]. As is well known, the Wick rotation enables one to compute the Feynman diagrams of perturbed QFT by means of Euclidean propagators. Let us suppose that it is not a technical trick, but quantum fields in an interaction zone are really Euclidean. It should be emphasized that the above mentioned Euclidean states differ from the well-known Schwinger functions in the Osterwalder–Shraded Euclidean QFT [7, 38, 49, 51, 55, 66]. The Schwinger functions are the Laplace transform of Wightman functions, but not chronological forms.

Since the chronological forms (10.2) are symmetric, the Euclidean states of a Borchers algebra A can be obtained as states of the corresponding commutative tensor algebra B_{RS^4} (7.1) [48, 49, 51]. Provided with the direct sum topology, B_{RS^4} becomes a topological involutive algebra. It coincides with the enveloping algebra of the Lie algebra of an additive Lie group $T(RS^4)$ of translations in RS^4. Therefore, one can obtain states of an algebra B_{RS^4} by constructing cyclic strongly continuous unitary representations of a nuclear Abelian group $T(RS^4)$ (Section 7). Such a representation is characterized by a continuous positive-definite generating function Z on SR^4. By virtue of Bochner Theorem 14.4, this function is the Fourier transform

$$Z(\phi) = \int \exp[i\langle\phi, y\rangle]d\mu(y) \tag{10.3}$$

of a positive measure μ of total mass 1 on the dual $(RS^4)'$ of RS^4. Then the above mentioned representation of $T(RS^4)$ can be given by operators

$$\widehat{\phi}\rho(y) = \exp[i\langle\phi, y\rangle]\rho(y) \tag{10.4}$$

in a Hilbert space $L^2_{\mathbb{C}}((RS^4)', \mu)$ of the equivalence classes of square μ-integrable complex functions $\rho(y)$ on $(RS^4)'$. A cyclic vector θ of this representation is the μ-equivalence class $\theta \approx_\mu 1$ of the constant function $\rho(y) = 1$.

Conversely, every positive measure μ of total mass 1 on the dual $(RS^4)'$ of RS^4 defines the cyclic strongly continuous unitary representation (10.4) of a group $T(RS^4)$. Herewith, distinct generating functions Z and Z' characterize equivalent representations T_Z and $T_{Z'}$ (10.4) of $T(RS^4)$ in the Hilbert spaces $L^2_{\mathbb{C}}((RS^4)', \mu)$ and $L^2_{\mathbb{C}}((RS^4)', \mu')$ iff they are the Fourier transform of equivalent measures on $(RS^4)'$ (Theorem 7.1).

If a generating function Z obeys the analiticity condition in Remark 7.3, a state f of B_{RS^4} is given by the expression

$$f_k(\phi_1 \cdots \phi_k) = i^{-k}\frac{\partial}{\partial\alpha^1}\cdots\frac{\partial}{\partial\alpha^k}Z(\alpha^i\phi_i)|_{\alpha^i=0} = \int \langle\phi_1, y\rangle \cdots \langle\phi_k, y\rangle d\mu(y). \tag{10.5}$$

Then one can think of Z (10.3) as being a generating functional of complete Euclidean Green functions f_k (10.5).

For instance, free Euclidean fields are described by Gaussian states. Their generating functions are of the form

$$Z(\phi) = \exp(-\frac{1}{2}M(\phi, \phi)), \tag{10.6}$$

where $M(\phi, \phi)$ is a positive-definite Hermitian bilinear form on RS^4 continuous in each variable. They are the Fourier transform of some Gaussian measure on $(RS^4)'$. In this case, the forms f_k (10.5) obey the Wick relations (8.16) where

$$f_1 = 0, \qquad f_2(\phi, \phi') = M(\phi, \phi').$$

Furthermore, a covariance form M on RS^4 is uniquely determined as

$$M(\phi_1, \phi_2) = \int W_2(x_1, x_2)\phi_1(x_1)\phi_2(x_2)d^n x_1 d^n x_2. \tag{10.7}$$

by a tempered distribution $W_2 \in S'(\mathbb{R}^8)$.

In particular, let a tempered distribution $M(\phi, \phi')$ in the expression (10.7) be Green's function of some positive elliptic differential operator $\mathcal{E}$, i.e.,

$$\mathcal{E}_{x_1}W_2(x_1, x_2) = \delta(x_1 - x_2),$$

where δ is Dirac's δ-function. Then the distribution W_2 reads

$$W_2(x_1, x_2) = w(x_1 - x_2), \tag{10.8}$$

and we obtain a form

$$f_2(\phi_1\phi_2) = M(\phi_1, \phi_2) = \int w(x_1 - x_2)\phi_1(x_1)\phi_2(x_2)d^4x_1d^4x_2 =$$
$$\int w(x)\phi_1(x_1)\phi_2(x_1 - x)d^4xd^4x_1 = \int w(x)\varphi(x)d^4x = \int w^F(p)\varphi^F(-p)d_4p,$$
$$x = x_1 - x_2, \qquad \varphi(x) = \int \phi_1(x_1)\phi_2(x_1 - x)d^4x_1.$$

For instance, if

$$\mathcal{E}_{x_1} = -\Delta_{x_1} + m^2,$$

where Δ is the Laplacian, then

$$w(x_1 - x_2) = \int \frac{\exp(-iq(x_1 - x_2))}{p^2 + m^2} d_4p, \tag{10.9}$$

where p^2 is the Euclidean square, is the propagator of a massive Euclidean scalar field. Note that, restricted to the domain $(x_1^0 - x_2^0) < 0$, it coincides with the Schwinger function $s_2(x_1 - x_2)$.

Let w^F be the Fourier transform of the distribution w (10.8). Then its Wick rotation is the functional

$$\widetilde{w}(x) = \theta(x)\int\limits_{Q_+} w^F(p)\exp(-px)d_4p + \theta(-x)\int\limits_{Q_-} w^F(p)\exp(-px)d_4p$$

on scalar fields on a Minkowski space [49, 51]. For instance, let $w(x)$ be the Euclidean propagator (10.9) of a massive scalar field. Then due to the analyticity of

$$w^F(p) = (p^2 + m^2)^{-1}$$

on the domain $\mathrm{Im}\, p \cdot \mathrm{Re}\, p > 0$, one can show that $\widetilde{w}(x) = -iD^c(x)$ where $D^c(x)$ is familiar causal Green's function.

A problem is that a measure μ in the generating function Z (10.3) fails to be written in an explicit form.

At the same time, a measure μ on $(RS^4)'$ is uniquely defined by a set of measures μ_N on the finite-dimensional spaces $\mathbb{R}_N = (RS^4)'/E$ where $E \subset (RS^4)'$ denotes a subspace of forms on RS^4 which are equal to zero at some finite-dimensional subspace $\mathbb{R}^N \subset RS^4$. The measures μ_N are images of μ under the canonical mapping $(RS^4)' \to \mathbb{R}_N$. For instance, every vacuum expectation $f_n(\phi_1 \cdots \phi_n)$ (10.5) admits the representation by an integral

$$f_k(\phi_1 \cdots \phi_k) = \int \langle w, \phi_1\rangle \cdots \langle w, \phi_k\rangle d\mu_N(w) \tag{10.10}$$

for any finite-dimensional subspace $\mathbb{R}^N$ which contains $\phi_1, \ldots, \phi_k$. In particular, one can replace the generating function (10.3) by the generating function

$$Z_N(\lambda_i e^i) = \int \exp(i\lambda_i w^i)\mu_N(w^i)$$

on $\mathbb{R}^N$ where $\{e^i\}$ is a basis for $\mathbb{R}^N$ and $\{w^i\}$ are coordinates with respect to the dual basis for $\mathbb{R}_N$. If f is a Gaussian state, we have the familiar expression (14.29):

$$d\mu_N = (2\pi \det[M^{ij}])^{-N/2} \exp[-\frac{1}{2}(M^{-1})_{ij} w^i w^j] d^N w, \tag{10.11}$$

where $M^{ij} = M(e^i, e^j)$ is a non-degenerate covariance matrix.

The representation (10.10) however is not unique, and the measure μ_N depends on the specification of a finite-dimensional subspace $\mathbb{R}^N$ of RS^4.

Remark 10.1. Note that an expression

$$\exp(-\int L(\phi) d^4x) \prod_x [d\phi(x)] \tag{10.12}$$

conventionally used in perturbed QFT is a symbolic functional integral, but not a true measure [23, 30, 56]. In particular, it is translationally invariant, i.e.,

$$[d\phi(x)] = [d(\phi(x) + \text{const.})],$$

whereas there is no (translationally invariant) Lebesgue measure on infinite-dimensional vector space as a rule (see [63] for an example of such a measure). $\diamondsuit$

11 Higgs vacuum

In contrast to the formal expression (10.12) of perturbed QFT, the true integral representation (10.3) of generating functionals enables us to handle non-Gaussian and inequivalent Gaussian representations of the commutative tensor algebra B_Q (7.1) of Euclidean scalar fields. Here, we describe one of such a representation as a model of a Higgs vacuum [47, 48].

In Standard Model of particle physics, Higgs vacuum is represented as a constant background part σ_0 of a Higgs scalar field σ [36, 58]. In algebraic QFT, one can describe free Higgs field similar to matter fields by a commutative tensor algebra B_Σ where Σ is a real nuclear space.

Let $Z(\widehat{\sigma})$ be the generating function (10.6) of a Gaussian state of B_Σ, and let μ be the corresponding Gaussian measure on the dual Σ' of Σ. In contrast with a finite-dimensional case, Gaussian measures on infinite-dimensional spaces fail to be quasi-invariant under translations as a rule. The introduction of a Higgs vacuum means a translation

$$\gamma : \Sigma' \ni \sigma \to \sigma + \sigma_0 \in \Sigma', \qquad \sigma_0 \in \Sigma'$$

in a space Σ' such that an original Gaussian measure $\mu(\sigma)$ is replaced by a measure $\mu_{\sigma_0}(\sigma) = \mu(\sigma + \sigma_0)$ possessing the Fourier transform

$$Z_{\sigma_0}(\widehat{\sigma}) = \exp(i\langle\widehat{\sigma}, \sigma_0\rangle Z(\widehat{\sigma})$$

The measures μ and μ_{σ_0} are equivalent iff a vector $\sigma_0 \in \Sigma'$ belongs to the canonical image of Σ in Σ' with respect to the scalar form $\langle|\rangle = M(,)$ (Remark 8.2). Then the measures μ and μ_{σ_0} define the equivalent states (10.5) of an algebra B_Σ. This equivalence is performed by the unitary operator

$$\rho(\sigma) \to \exp(-\langle\sigma|\sigma_0\rangle)\rho(\sigma + \sigma_0), \qquad \rho(\sigma) \in L^2(\Sigma', \mu).$$

This operator fails to be constructed if $\sigma_0 \in \Sigma' \setminus \Sigma$, and the measures μ and μ_{σ_0} are inequivalent.

Following the terminology of Standard Model, let us call $\sigma_0 \in \Sigma' \setminus \Sigma$ the Higgs vacuum field and $\sigma \in \Sigma \subset \Sigma'$ the Higgs boson fields. Then we can say the following.

(i) A Higgs vacuum field σ_0 and Higgs boson fields σ belong to different classes of functions. For instance, one usually chooses a constant Higgs vacuum field σ_0 in QFT. If $\Sigma = RS^4$, a constant function is an element of $(SR^4)' \setminus SR^4$. At the same time, since Σ is dense in Σ', the elements σ_0 and σ can be arbitrarily closed to each other with respect to a topology in Σ'. However, a covariance form M and some other functions being well defined at points σ become singular at points σ_0.

(ii) One can think of a Higgs vacuum field σ_0 as being the classical one in the sense that $\sigma_0 \in \Sigma' \setminus \Sigma$, whereas Higgs boson fields $\sigma \in \Sigma \subset \Sigma'$ are quantized fields because the possess quantum partners $\widehat{\sigma} \in \Sigma \subset B_\Sigma$.

(iii) States of Higgs boson fields in the presence of in equivalent Higgs vacua are inequivalent.

(iv) Let the generating function Z and Z_{σ_0} be restricted to some finite-dimensional subspace $\mathbb{R}^N \subset \Sigma$. Then there exists an element $\sigma_{0N} \in \mathbb{R}_N$ such that $\langle \widehat{\sigma}, \sigma_0 \rangle = \langle \widehat{\sigma} | \sigma_{0N} \rangle$ for any $\widehat{\sigma} \in \mathbb{R}^N$. As a consequence, a generating function Z_{σ_0} takes a form

$$Z_{\sigma_0 N}(\lambda_i \widehat{\sigma}^i) = (2\pi \det[M^{ij}])^{-N/2} \int \exp(\lambda_i \sigma^i) \exp[-\frac{1}{2}(M^{-1})_{ij}(\sigma - \sigma_{0N})^i (\sigma - \sigma_{0N})^j] d^N \sigma,$$

where M^{ij} is the covariance matrix of Z_N, σ^i denote coordinates in $\mathbb{R}_N$, and σ^i_{0N} are coordinates of a vector σ_{0N} in $\mathbb{R}_N$. It follows that, if a number of quantum Higgs boson fields $wh\sigma$ is finite, their interaction with a classical Higgs vacuum field σ_0 reduced to an interaction with some quantum fields $\widehat{\sigma}_{0N}$ by perturbation theory.

In Standard Model, a Higgs vacuum is responsible for spontaneous symmetry breaking. Let us study this phenomenon (Sections 12 and 13).

12 Automorphisms of quantum systems

In order to say something, we mainly restrict our analysis to automorphisms of C^* algebras.

We consider uniformly and strongly continuous one-parameter groups of automorphisms of C^*-algebras. Let us note that any weakly continuous one-parameter group of endomorphism of a C^*-algebra also is also strongly continuous and their weak and strong generators coincide with each other [12, 24].

Remark 12.1. There is the following relation between morphisms of a C^*-algebra A and a set $F(A)$ of its states which is a convex subset of the dual A' of A (Theorem 3.10). A linear morphism γ of a C^*-algebra A as a vector space is called the Jordan morphism if relations

$$\gamma(ab + ba) = \gamma(a)\gamma(b) + \gamma(b)\gamma(a), \qquad \phi(a^*) = \gamma(a)^*, \qquad a, b \in A.$$

hold. One can show the following [20]. Let γ be a Jordan automorphism of a unital C^*-algebra A. It yields the dual weakly* continuous affine bijection γ' of $F(A)$ onto itself, i.e.,

$$\gamma'(\lambda f + (1-\lambda)f') = \lambda\gamma'(f) + (1-\lambda)\gamma'(f'), \qquad f, f', \in F(A), \qquad \lambda \in [0,1].$$

Conversely, any such a map of $F(A)$ is the dual to some Jordan automorphism of A. However, if G is a connected group of weakly continuous Jordan automorphisms of a unital C^*-algebra A is a weakly (and, consequently, strongly) continuous group of automorphisms of A. $\diamondsuit$

A topological group G is called the strongly (resp. uniformly) continuous group of automorphisms of a C^*-algebra A if there is its continuous monomorphism to the group $\mathrm{Aut}\,(A)$ of automorphisms of A provided with the strong (resp. normed) operator topology, and if its action on A is separately continuous.

One usually deals with strongly continuous groups of automorphisms because of the following reason. Let $G(\mathbb{R})$ be one-parameter group of automorphisms of a C^*-algebra A. This group is uniformly (resp. strongly) continuous if it is a range of a continuous map of $\mathbb{R}$ to the group $\mathrm{Aut}\,(A)$ of automorphisms of A which is provided with the normed (resp. strong) operator topology and whose action on A is separately continuous. A problem is that, if a curve $G(\mathbb{R})$ in $\mathrm{Aut}\,(A)$ is continuous with respect to the normed operator topology, then a curve $G(\mathbb{R})(a)$ for any $a \in A$ is continuous in a C^*-algebra A, but the converse is not true. At the same time, a curve $G(\mathbb{R})$ is continuous in $\mathrm{Aut}\,(A)$ with respect to the strong operator topology iff a curve $G(\mathbb{R})(a)$ for any $a \in A$ is continuous in A. By this reason, strongly continuous one-parameter groups of automorphisms of C^*-algebras are most interesting. However, the infinitesimal generator of such a group fails to be bounded, unless this group is uniformly continuous.

Remark 12.2. If $G(\mathbb{R})$ is a strongly continuous one-parameter group of automorphisms of a C^*-algebra A, there are the following continuous maps [12]:

- $\mathbb{R} \ni t \to \langle G_t(a), f\rangle \in \mathbb{C}$ is continuous for all $a \in A$ and $f \in A'$;
- $A \ni a \to G_t(a) \in A$ is continuous for all $t \in \mathbb{R}$;
- $\mathbb{R} \ni t \to G_t(a) \in A$ is continuous for all $a \in A$. $\diamondsuit$

Without a loss of generality, we further assume that A is a unital C^*-algebra. Infinitesimal generators of one-parameter groups of automorphisms of A are derivations of A.

By a derivation δ of A throughout is meant an (unbounded) symmetric derivation of A (i.e., $\delta(a^*) = \delta(a)^*$, $a \in A$) which is defined on a dense involutive subalgebra $D(\delta)$ of A. If a derivation δ on $D(\delta)$ is bounded, it is extended to a bounded derivation everywhere on A. Conversely, every derivation defined everywhere on a C^*-algebra is bounded [18]. For instance, any inner derivation $\delta(a) = i[b, a]$, where b is a Hermitian element of A, is bounded. A space of derivations of A is provided with the involution $u \to u^*$ defined by the equality

$$\delta^*(a) = -\delta(a^*)^*, \qquad a \in \mathcal{A}. \tag{12.1}$$

There is the following relation between bounded derivations of a C^*-algebra A and uniformly continuous one-parameter groups of automorphisms of A [12].

Theorem 12.1. Let δ be a derivation of a C^*-algebra A. The following assertions are equivalent:

- δ is defined everywhere and, consequently, is bounded;
- δ is the infinitesimal generator of a uniformly continuous one-parameter group $G(\mathbb{R})$ of automorphisms of a C^*-algebra A.

Furthermore, for any representation π of A in a Hilbert space E, there exists a bounded self-adjoint operator $\mathcal{H} \in \pi(A)''$ in E and the unitary uniformly continuous representation

$$\pi(G_t) = \exp(-it\mathcal{H}), \qquad t \in \mathbb{R}, \tag{12.2}$$

of the group $G(\mathbb{R})$ in E such that

$$\pi(\delta(a)) = -i[\mathcal{H}, \pi(a)], \qquad a \in A, \tag{12.3}$$

$$\pi(G_t(a)) = e^{-it\mathcal{H}}\pi(a)e^{it\mathcal{H}}, \qquad t \in \mathbb{R}. \tag{12.4}$$

$\square$

A C^*-algebra need not admit non-zero bounded derivations. For instance, no commutative C^*-algebra possesses bounded derivations. The following is the relation between (unbounded) derivations of a C^*-algebra A and strongly continuous one-parameter groups of automorphisms of A [11, 43].

Theorem 12.2. Let δ be a closable derivation of a C^*-algebra A. Its closure $\overline{\delta}$ is an infinitesimal generator of a strongly continuous one-parameter group of automorphisms of A iff

(i) the set $(\mathbf{1} + \lambda\delta)(D(\delta)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$ is dense in A,

(ii) $\|(\mathbf{1} + \lambda\delta)(a)\| \geq \|a\|$ for any $\lambda \in \mathbb{R}$ and any $a \in A$. $\square$

It should be noted that, if A is a unital algebra and δ is its closable derivation, then $\mathbf{1} \in D(\delta)$.

Let us mention a more convenient sufficient condition of a derivation of a C^*-algebra to be an infinitesimal generator of a strongly continuous one-parameter group of its automorphisms. A derivation δ of a C^*-algebra A is called well-behaved if, for each element $a \in D(\delta)$, there exists a state f of A such that $f(a) = \|a\|$ and $f(\delta(a)) = 0$. If δ is a well-behaved derivation, it is closable [33], and obeys the condition (ii) in Theorem 12.2 [11, 43]. Then we come to the following.

Theorem 12.3. If δ is a well-behaved derivation of a C^*-algebra A and it obeys the condition (i) in Theorem 12.2, its closure $\overline{\delta}$ is an infinitesimal generator of a strongly continuous one-parameter group of automorphisms of A. $\square$

For instance, a derivation δ is well-behaved if it is approximately inner, i.e., there exists a sequence of self-adjoint elements $\{b_n\}$ in A such that

$$\delta(a) = \lim_n i[b_n, a], \qquad a \in A.$$

In contrast with a case of a uniformly continuous one-parameter group of automorphisms of a C^*-algebra A, a representation of A does not imply necessarily a unitary representation (12.2) of a strongly continuous one-parameter group of automorphisms of A, unless the following.

Theorem 12.4. Let $G(\mathbb{R})$ be a strongly continuous one-parameter group of automorphisms of a C^*-algebra A and δ its infinitesimal generator. Let A admit a state f such that

$$|f(\delta(a))| \leq \lambda[f(a^*a) + f(aa^*)]^{1/2} \tag{12.5}$$

for all $a \in A$ and a positive number λ, and let (π_f, θ_f) be a cyclic representation of A in a Hilbert space E_f defined by f. Then there exist a self-adjoint operator $\mathcal{H}$ on a domain $D(\mathcal{H}) \subset A\theta_f$ in E_f and the unitary strongly continuous representation (12.2) of $G(\mathbb{R})$ in E_f which fulfils the relations (12.3) – (12.4) for $\pi = \pi_f$. $\square$

It should be emphasized that the condition (12.5) in Theorem 12.4 is sufficient in order that a derivation δ to be closable [33].

Note that there is a general problem of a unitary representation of an automorphism group of a C^*-algebra A. Let π be a representation of A in a Hilbert space E. Then an automorphism ρ of A possesses a unitary representation in E if there exists a unitary operator U_ρ in E such that

$$\pi(\rho(a)) = U_\rho \pi(a) U_\rho^{-1}, \qquad a \in A. \tag{12.6}$$

A key point is that such a representation is never unique. Namely, let U and U' be arbitrary unitary elements of the commutant $\pi(A)'$ of $\pi(A)$. Then $UU_\rho U'$ also provides a unitary representation of ρ. For instance, one can always choose phase multipliers $U = \exp(i\alpha)\mathbf{1} \in U(1)$. A consequence of this ambiguity is the following.

Let G be a group of automorphisms of an algebra A whose elements $g \in G$ admit unitary representations U_g (12.6). The set of operators U_g, $g \in G$, however need not be a group. In general, we have

$$U_g U_{g'} = U(g,g')U_{gg'}U'(g,g'), \qquad U(g,g'), U'(g,g') \in \pi(A)'.$$

If all $U(g,g')$ are phase multipliers, one says that the unitary operators U_g, $g \in G$, form a projective representation $U(G)$:

$$U_g U_{g'} = k(g,g')U_{gg'}, \qquad g, g' \in G,$$

of a group G [16, 62]. In this case, a set $U(1) \times U(G)$ becomes a group which is a central $U(1)$-extension

$$\mathbf{1} \longrightarrow U(1) \longrightarrow U(1) \times U(G) \longrightarrow G \longrightarrow \mathbf{1} \tag{12.7}$$

of a group G. Accordingly, the projective representation $\pi(G)$ of G is a splitting of the exact sequence (12.7). It is characterized by $U(1)$-multipliers $k(g,g')$ which form a two-cocycle

$$k(\mathbf{1},g) = k(g,\mathbf{1}) = \mathbf{1}, \qquad k(g_1, g_2g_3)k(g_2,g_3) = k(g_1,g_2)k(g_1g_2,g_3) \tag{12.8}$$

of the cochain complex of G with coefficients in $U(1)$ [24, 34]. A different splitting of the exact sequence (12.7) yields a different projective representation $U'(G)$ of G whose multipliers $k'(g,g')$ form a cocycle equivalent to the cocycle (12.8). If this cocycle is a coboundary, there exists a splitting of the extension (12.7) which provides a unitary representation of a group G of automorphisms of an algebra A in E.

For instance, let $B(E)$ be a C^*-algebra of bounded operators in a Hilbert space E. All its automorphisms are inner. Any (unitary) automorphism U of a Hilbert space E yields an inner automorphism

$$a \to UaU^{-1}, \qquad a \in B(E), \tag{12.9}$$

of $B(E)$. Herewith, the automorphism (12.9) is the identity iff $U = \lambda\mathbf{1}$, $|\lambda| = 1$, is a scalar operator in E. It follows that the group of automorphisms of $B(E)$ is the quotient $U(E)/U(1)$ of a unitary group $U(E)$ with respect to a circle subgroup $U(1)$. Therefore, given a group G of automorphisms of the C^*-algebra $B(E)$, the representatives U_g in $U(E)$ of elements $g \in G$ constitute a group up to phase multipliers, i.e.,

$$U_g U_{g'} = \exp[i\alpha(g,g')]U_{gg'}, \qquad \alpha(g,g') \in \mathbb{R}.$$

Nevertheless, if G is a one-parameter weakly* continuous group of automorphisms of $B(E)$ whose infinitesimal generator is a bounded derivation of $B(E)$, one can choose the multipliers $\exp[i\alpha(g,g')] = 1$.

In a general setting, let G be a group and $\mathcal{A}$ a commutative algebra. An $\mathcal{A}$-multiplier of G is a map $\xi : G \times G \to \mathcal{A}$ such that

$$\xi(\mathbf{1}_G, g) = \xi(g, \mathbf{1}_G) = \mathbf{1}_\mathcal{A}, \qquad \xi(g_1, g_2g_3)\xi(g_2,g_3) = \xi(g_1,g_2)\xi(g_1g_2,g_3), \qquad g, g_i \in G.$$

For instance, $\xi : G \times G \to \mathbf{1}_{\mathcal{A}} \in \mathcal{A}$ is a multiplier. Two A-multipliers ξ and ξ' are said to be equivalent if there exists a map $f : G \to \mathcal{A}$ such that

$$\xi(g_1, g_2) = \frac{f(g_1 g_2)}{f(g_1) f(g_2)} \xi'(g_1, g_2), \qquad g_i \in G.$$

An $\mathcal{A}$-multiplier is called exact if it is equivalent to the multiplier $\xi = \mathbf{1}_{\mathcal{A}}$. A set of $\mathcal{A}$-multipliers is an Abelian group with respect to the pointwise multiplication, and the set of exact multipliers is its subgroup. Let $HM(G, \mathcal{A})$ be the corresponding factor group.

If G is a locally compact topological group and $\mathcal{A}$ a Hausdorff topological algebra, one additionally requires that multipliers ξ and equivalence maps f are measurable maps. In this case, there is a natural topology on $HM(G, \mathcal{A})$ which is locally quasi-compact, but need not be Hausdorff [35].

Theorem 12.5. [16]. Let G be a simply connected locally compact Lie group. Each $U(1)$-multiplier ξ of G is brought into a form $\xi = \exp i\alpha$, where α is an $\mathbb{R}$-multiplier. Moreover, ξ is exact iff α is well. Any $\mathbb{R}$-multiplier of G is equivalent to a smooth one. □

Let G be a locally compact group of strongly continuous automorphisms of a C^*-algebra A. Let $M(A)$ denote a multiplier algebra of A, i.e., the largest C^*-algebra containing A as an essential ideal, i.e., if $a \in M(A)$ and $ab = 0$ for all $b \in A$, then $a = 0$ [65]). For instance, $M(A) = A$ if A is a unital algebra. Let ξ be a multiplier of G with values in the center of $M(A)$. A G-covariant representation π of A [19, 37] is a representation π of A (and, consequently, $M(A)$) in a Hilbert space E together with a projective representation of G by unitary operators $U(g)$, $g \in G$, in E such that

$$\pi(g(a)) = U(g)\pi(a)U^*(g), \qquad U(g)U(g') = \pi(\xi(g, g'))U(gg').$$

13 Spontaneous symmetry breaking

Given a topological *-algebra A and its state f, let ρ be an automorphism of A. Then it defines a state

$$f_\rho(a) = f(\rho(a)), \qquad a \in A, \tag{13.1}$$

of A. A state f is said to be stationary with respect to an automorphism ρ of A if

$$f(\rho(a)) = f(a), \qquad a \in A. \tag{13.2}$$

One speaks about spontaneous symmetry breaking if a state f of a quantum algebra A fails to be stationary with respect to some automorphisms of A.

We can say something if A is a C^*-algebra and its GNS representations are considered [12, 18, 24].

Theorem 13.1. Let f be a state of a C^*-algebra A and (π_f, θ_f, E_f) the corresponding cyclic representation of A. An automorphism ρ of A defines the state f_ρ (13.1) of A such that a carrier space $E_{\rho f}$ of the corresponding cyclic representation $\pi_{\rho f}$ is isomorphic to E_f. □

It follows that the representations $\pi_{\rho f}$ can be given by operators $\pi_{\rho f}(a) = \pi_f(\rho(a))$ in the carrier space E_f of the representation π_f, but these representations fail to be equivalent, unless

an automorphism ρ possesses the unitary representation (12.6) in E_f. In this case, a state f is stationary relative to ρ. The converse also is true.

Theorem 13.2. If a state f of a C^*-algebra A is the stationary state (13.2) with respect to an automorphism ρ of A, there exists a unique unitary representation U_ρ (13.1) of ρ in E_f such that

$$U_\rho \theta_f = \theta_f. \tag{13.3}$$

□

It follows from Theorem 12.1 that, since any uniformly continuous one-parameter group of automorphisms of a C^*-algebra A admits a unitary representation, each state f of A is stationary for this group. However, this is not true for an arbitrary uniformly continuous group G of automorphisms of A. For instance, let $B(E)$ be the C^*-algebra of all bounded operators in a Hilbert space E. Any automorphisms of $B(E)$ is inner and, consequently, possesses a unitary representation in E. Since the commutant of $B(E)$ reduces to scalars, the group of automorphisms of $B(E)$ admits a projective representation in E, but it need not be unitary.

It follows from Theorem 12.4) that, if a state f of a C^*-algebra A is stationary under a strongly continuous group $G(\mathbb{R})$ of automorphisms of A, i.e., $f(]dl(a)) = 0$, there exists unitary representation of this group in E_f. However, this condition is sufficient, but not necessary.

Moreover, one can show the following [18].

Theorem 13.3. Let G be a strongly continuous group of automorphisms of a C^*-algebra A, and let a state f of A be stationary for G. Then there exists a unique unitary representation of G in E_f whose operators obey the equality (13.3). □

Let now G be a group of strongly or uniformly continuous group of automorphisms of a C^*-algebra A, and let f be a state of A. Let us consider a set of states f_g (13.1), $g \in G$, of A defined by automorphisms $g \in G$. Let f be stationary with respect to a proper subgroup H of G. Then a set of equivalence classes of states f_g, $g \in G$, is a subset of the factor space G/H, but need not coincide with G/H.

This is just the case of spontaneous symmetry breaking in Standard Model where A Higgs vacuum is a stationary state with respect to some proper subgroup of a symmetry group [36, 58].

In axiomatic QFT, the spontaneous symmetry breaking phenomenon is described by the Goldstone theorem [7].

Let G be a connected Lie group of internal symmetries (automorphisms of the Borchers algebra A over $\mathrm{Id}\,\mathbb{R}^4$) whose infinitesimal generators are given by conserved currents j^k_μ. One can show the following [7].

Theorem 13.4. A group G of internal symmetries possesses a unitary representation in E_W iff the Wightman functions are G-invariant. □

Theorem 13.5. A group G of internal symmetries admits a unitary representation if a strong spectrum condition holds, i.e., there exists a mass gap. □

As a consequence, we come to the above mentioned Goldstone theorem.

Theorem 13.6. If there is a group G of internal symmetries which are spontaneously broken, there exist elements $\phi \in E_W$ of zero spin and mass such that $\langle \phi | j^k_\mu \psi_0 \rangle \neq 0$ for some generators of G. □

These elements of unit norm are called Goldstone states. It is easily observed that, if a group G of spontaneously broken symmetries contains a subgroup of exact symmetries H, the Goldstone states carrier out a homogeneous representation of G isomorphic to the quotient G/H.

This fact attracted great attention to such kind representations and motivated to describe classical Higgs fields as sections of a fibre bundle with a typical fibre G/H [52, 53, 54].

14 Appendixes

This Section summarizes some relevant material on topological vector spaces and measures on non-compact spaces.

14.1 Topological vector spaces

There are several standard topologies introduced on an (infinite-dimensional) complex or real vector space and its dual [24, 44]. Topological vector spaces throughout are assumed to be locally convex. Unless otherwise stated, by the dual V' of a topological vector space V is meant its topological dual, i.e., the space of continuous linear maps of $V \to \mathbb{R}$.

Let us note that a topology on a vector space V often is determined by a set of seminorms. A non-negative real function p on V is called the seminorm if it satisfies the conditions

$$p(\lambda x) = |\lambda| p(x), \qquad p(x+y) \leq p(x) + p(y), \qquad x, y \in V, \quad \lambda \in \mathbb{R}.$$

A seminorm p for which $p(x) = 0$ implies $x = 0$ is called the norm. Given any set $\{p_i\}_{i\in I}$ of seminorms on a vector space V, there is the coarsest topology on V compatible with the algebraic structure such that all seminorms p_i are continuous. It is a locally convex topology whose base of closed neighborhoods consists of sets

$$\{x : \sup_{1\leq i\leq n} p_i(x) \leq \varepsilon\}, \qquad \varepsilon > 0, \qquad n \in \mathbb{N}_+.$$

Let V and W be two vector spaces whose Cartesian product $V \times W$ is provided with a bilinear form $\langle v, w\rangle$ which obeys the following conditions:

- for any element $v \neq 0$ of V, there exists an element $w \in W$ such that $\langle v, w\rangle \neq 0$;
- for any element $w \neq 0$ of W, there exists an element $v \in V$ such that $\langle v, w\rangle \neq 0$.

Then one says that (V, W) is a dual pair. If (V, W) is a dual pair, so is (W, V). Clearly, W is isomorphic to a vector subbundle of the algebraic dual V^* of V, and V is a subbundle of the algebraic dual of W.

Given a dual pair (V, W), every vector $w \in W$ defines a seminorm $p_w = |\langle v, w\rangle|$ on V. The coarsest topology $\sigma(V, W)$ on V making all these seminorms continuous is called the weak topology determined by W on V. It also is the coarsest topology on V such that all linear forms in $W \subset V^*$ are continuous. Moreover, W coincides with the dual V' of V provided with the weak topology $\sigma(V, W)$, and $\sigma(V, W)$ is the coarsest topology on V such that $V' = W$. Of course, the weak topology is Hausdorff.

For instance, if V is a Hausdorff topological vector space with the dual V', then (V, V') is a dual pair. The weak topology $\sigma(V, V')$ on V is coarser than the original topology on V. Since (V', V) also is a dual pair, the dual V' of V can be provided with the weak* topology topology $\sigma(V', V)$. Then V is the dual of V', equipped with the weak* topology.

The weak* topology is the coarsest case of a topology of uniform convergence on V'. A subset M of a vector space V is said to absorb a subset $N \subset V$ if there is a number $\epsilon \geq 0$ such that $N \subset \lambda M$ for all λ with $|\lambda| \geq \epsilon$. An absorbent set is one which absorbs all points. A subset N of a topological vector space V is called bounded if it is absorbed by any neighborhood of the origin of V. Let (V, V') be a dual pair and $\mathcal{N}$ some family of weakly bounded subsets of V. Every $N \subset \mathcal{N}$ yields a seminorm

$$p_N(v') = \sup_{v \in N} |\langle v, v' \rangle|$$

on the dual V' of V. A topology on V' defined by a set of seminorms p_N, $N \in \mathcal{N}$, is called the topology of uniform convergence on the sets of $\mathcal{N}$. When $\mathcal{N}$ is a set of all finite subsets of V, we have the coarsest topology of uniform convergence which is the above mentioned weak* topology $\sigma(V', V)$. The finest topology of uniform convergence is obtained by taking $\mathcal{N}$ to be a set of all weakly bounded subsets of V. It is called the strong topology. The dual V'' of V', provided with the strong topology, is called the bidual. One says that V is reflexive if $V = V''$.

Since (V', V) is a dual pair, a vector space V also can be provided with the topology of uniform convergence on subsets of V', e.g., the weak* and strong topologies. Moreover, any Hausdorff locally convex topology on V is a topology of uniform convergence. The coarsest and finest topologies of them are the weak* and strong topologies, respectively. There is the following chain

$$\text{weak}^* < \text{weak} < \text{original} < \text{strong}$$

of topologies on V, where $<$ means "to be finer".

For instance, let V be a normed space. The dual V' of V also is equipped with a norm

$$\|v'\|' = \sup_{\|v\|=1} |\langle v, v' \rangle|, \qquad v \in V, \qquad v' \in V'. \tag{14.1}$$

Let us consider a set of all balls $\{v : \|v\| \leq \epsilon, \, \epsilon > 0\}$ in V. The topology of uniform convergence on this set coincides with strong and normed topologies on V' because weakly bounded subsets of V also are bounded by a norm. Normed and strong topologies on V are equivalent. Let $\overline{V}$ denote the completion of a normed space V. Then V' is canonically identified to $(\overline{V})'$ as a normed space, though weak* topologies on V' and $(\overline{V})'$ are different. Let us note that both V' and V'' are Banach spaces. If V is a Banach space, it is closed in V'' with respect to the strong topology on V'' and dense in V'' equipped with the weak* topology. One usually considers the weak*, weak and normed (equivalently, strong) topologies on a Banach space.

It should be noted that topology on a finite-dimensional vector space is locally convex and Hausdorff iff it is determined by the Euclidean norm.

Let us say a few words on morphisms of topological vector spaces.

A linear morphism between two topological vector spaces is called the weakly continuous morphism if it is continuous with respect to the weak topologies on these vector spaces. In particular, any continuous morphism between topological vector spaces is weakly continuous [44].

A linear morphism between two topological vector spaces is called bounded if the image of a bounded set is bounded. Any continuous morphism is bounded. A topological vector space is called the Mackey space if any bounded endomorphism of this space is continuous (we follow the terminology of [44]). Metrizable and, consequently, normed spaces are of this type.

Any linear morphism $\gamma : V \to W$ of topological vector spaces yields the dual morphism $\gamma' : W' \to V'$ of the their topological duals such that

$$\langle v, \gamma'(w)\rangle = \langle \gamma(v), w\rangle, \qquad v \in V, \qquad w \in W.$$

If γ is weakly continuous, then γ' is weakly* continuous. If V and W are normed spaces, then any weakly continuous morphism $\gamma : V \to W$ is continuous and strongly continuous. Given normed topologies on V' and W', the dual morphism $\gamma' : W' \to V'$ is continuous iff γ is continuous.

14.2 Hilbert, countably Hilbert and nuclear spaces

Let us recall the relevant basics on pre-Hilbert and Hilbert spaces [10, 24].

A Hermitian form on a complex vector space E is defined as a sesquilinear form $\langle .|.\rangle$ such that

$$\langle e|e'\rangle = \overline{\langle e'|e\rangle}, \qquad \langle \lambda e|e'\rangle = \langle e|\overline{\lambda}e'\rangle = \lambda\langle e|e'\rangle, \quad e, e' \in E, \quad \lambda \in \mathbb{C}.$$

Remark 14.1. There exists another convention where $\langle e|\lambda e'\rangle = \lambda\langle e|e'\rangle$. $\diamondsuit$

A Hermitian form $\langle .|.\rangle$ is said to be positive if $\langle e|e\rangle \geq 0$ for all $e \in E$. All Hermitian forms throughout are assumed to be positive. A Hermitian form is called non-degenerate if the equality $\langle e|e\rangle = 0$ implies $e = 0$. A complex vector space endowed with a Hermitian form is called the pre-Hilbert space. Morphisms of pre-Hilbert spaces, by definition, are isometric.

A Hermitian form provides E with the topology defined by a seminorm $\|e\| = \langle e|e\rangle^{1/2}$. Hence, a pre-Hilbert space is Hausdorff iff a Hermitian form $\langle .|.\rangle$ is non-degenerate, i.e., a seminorm $\|e\|$ is a norm. In this case, it is called the scalar product.

A complete Hausdorff pre-Hilbert space is called the Hilbert space. Any Hausdorff pre-Hilbert space can be completed to a Hilbert space. .

The following are the standard constructions of new Hilbert spaces from the old ones.

• Let $(E^\iota, \langle .|.\rangle_{E^\iota})$ be a set of Hilbert spaces and $\sum E^\iota$ denote a direct sum of vector spaces E^ι. For any two elements $e = (e^\iota)$ and $e' = (e'^\iota)$ of $\sum E^\iota$, a sum

$$\langle e|e'\rangle_\oplus = \sum_\iota \langle e^\iota|e'^\iota\rangle_{E^\iota} \tag{14.2}$$

is finite, and defines a non-degenerate Hermitian form on $\sum E^\iota$. The completion $\oplus E^\iota$ of $\sum E^\iota$ with respect to this form is a Hilbert space, called the Hilbert sum of E^ι.

• Let $(E, \langle .|.\rangle_E)$ and $(H, \langle .|.\rangle_H)$ be Hilbert spaces. Their tensor product $E \otimes H$ is defined as the completion of a tensor product of vector spaces E and H with respect to the scalar product

$$\langle w_1|w_2\rangle_\otimes = \sum_{\iota,\beta} \langle e_1^\iota|e_2^\beta\rangle_E \langle h_1^\iota|h_2^\beta\rangle_H,$$

$$w_1 = \sum_\iota e_1^\iota \otimes h_1^\iota, \quad w_2 = \sum_\beta e_2^\beta \otimes h_2^\beta, \quad e_1^\iota, e_2^\beta \in E, \quad h_1^\iota, h_2^\beta \in H.$$

• Let E' be the topological dual of a Hilbert space E. Then the assignment

$$e \to \overline{e}(e') = \langle e'|e\rangle, \qquad e, e' \in E, \tag{14.3}$$

defines an antilinear bijection of E onto E', i.e., $\overline{\lambda e} = \overline{\lambda}\overline{e}$. The dual E' of a Hilbert space is a Hilbert space provided with the scalar product $\langle \overline{e}|\overline{e}'\rangle' = \langle e'|e\rangle$ such that the morphism (14.3) is isometric. The E' is called the dual Hilbert space, and is denoted by $\overline{E}$.

Physical applications of Hilbert spaces are limited by the fact that the dual of a Hilbert space E is anti-isomorphic to E. The construction of a rigged Hilbert space describes the dual pairs (E, E') where E' is larger than E [22].

Let a complex vector space E have a countable set of non-degenerate Hermitian forms $\langle .|.\rangle_k$, $k \in \mathbb{N}_+$, such that

$$\langle e|e\rangle_1 \leq \cdots \leq \langle e|e\rangle_k \leq \cdots$$

for all $e \in E$. The family of norms

$$\|.\|_k = \langle .|.\rangle_k^{1/2}, \qquad k \in \mathbb{N}_+, \tag{14.4}$$

yields a Hausdorff topology on E. A space E is called the countably Hilbert space if it is complete with respect to this topology [22]. For instance, every Hilbert space is a countably Hilbert space where all Hermitian forms $\langle .|.\rangle_k$ coincide.

Let E_k denote the completion of E with respect to the norm $\|.\|_k$ (14.4). There is the chain of injections

$$E_1 \supset E_2 \supset \cdots E_k \supset \cdots \tag{14.5}$$

together with a homeomorphism $E = \underset{k}{\cap} E_k$. The dual spaces form the increasing chain

$$E'_1 \subset E'_2 \subset \cdots \subset E'_k \subset \cdots, \tag{14.6}$$

and $E' = \underset{k}{\cup} E'_k$. The dual E' of E can be provided with the weak* and strong topologies. One can show that a countably Hilbert space is reflexive.

Given a countably Hilbert space E and $m \leq n$, let T^n_m be a prolongation of the map

$$E_n \supset E \ni e \to e \in E \subset E_m$$

to a continuous map of E_n onto a dense subset of E_m. A countably Hilbert space E is called the nuclear space if, for any m, there exists n such that T^m_n is a nuclear map, i.e.,

$$T^n_m(e) = \sum_i \lambda_i \langle e|e^i_n\rangle_{E_n} e^i_m,$$

where: (i) $\{e^i_n\}$ and $\{e^i_m\}$ are bases for the Hilbert spaces E_n and E_m, respectively, (ii) $\lambda_i \geq 0$, (iii) the series $\sum \lambda_i$ converges [22].

An important property of nuclear spaces is that they are perfect, i.e., every bounded closed set in a nuclear space is compact. It follows immediately that a Banach (and Hilbert) space is not nuclear, unless it is finite-dimensional. Since a nuclear space is perfect, it is separable, and the weak* and strong topologies (and, consequently, all topologies of uniform convergence) on a nuclear space E and its dual E' coincide.

Let E be a nuclear space, provided with still another non-degenerate Hermitian form $\langle .|.\rangle$ which is separately continuous, i.e., continuous with respect to each argument. It follows that there exist numbers M and m such that

$$\langle e|e\rangle \leq M\|e\|_m, \qquad e \in E. \tag{14.7}$$

Let $\widetilde{E}$ denote the completion of E with respect to this form. There are the injections

$$E \subset \widetilde{E} \subset E', \tag{14.8}$$

where E is a dense subset of $\widetilde{E}$ and $\widetilde{E}$ is a dense subset of E', equipped with the weak* topology. The triple (14.8) is called the rigged Hilbert space. Furthermore, bearing in mind the chain of Hilbert spaces (14.5) and that of their duals (14.6), one can convert the triple (14.8) into the chain of spaces

$$E \subset \cdots \subset E_k \subset \cdots E_1 \subset \widetilde{E} \subset E'_1 \subset \cdots \subset E'_k \subset \cdots \subset E'. \tag{14.9}$$

Remark 14.2. Real Hilbert, countably Hilbert, nuclear and rigged Hilbert spaces are similarly described. $\diamondsuit$

14.3 Measures on locally compact spaces

Measures on a locally compact space X are defined as continuous forms on spaces of continuous real (or complex) functions of compact support on X, and they are extended to a wider class of functions on X [9, 24].

Let $\mathbb{C}^0(X)$ be a ring of continuous complex functions on X. For each compact subset K of X, we have a seminorm

$$p_K(f) = \sup_{x\in K} |f(x)| \tag{14.10}$$

on $C^0(X)$. These seminorms provide $\mathbb{C}^0(X)$ with the Hausdorff topology of compact convergence. We abbreviate with $\mathcal{K}(X,\mathbb{C})$ the dense subspace of $\mathbb{C}^0(X)$ which consists of continuous complex functions of compact support on X. It is a Banach space with respect to a norm

$$\|f\| = \sup_{x\in X} |f(x)|. \tag{14.11}$$

Its normed topology, called the topology of uniform convergence, is finer than the topology of compact convergence, and these topologies coincide if X is a compact space.

A space $\mathcal{K}(X,\mathbb{C})$ also can be equipped with another topology, which is especially relevant to integration theory. For each compact subset $K \subset X$, let $\mathcal{K}_K(X,\mathbb{C})$ be a vector subspace of $\mathcal{K}(X,\mathbb{C})$ consisting of functions of support in K. Let $\mathcal{U}$ be a set of all absolutely convex absorbent subsets U of $\mathcal{K}(X,\mathbb{C})$ such that, for every compact K, a set $U \cap \mathcal{K}_K(X,\mathbb{C})$ is a neighborhood of the origin in $\mathcal{K}_K(X,\mathbb{C})$ under the topology of uniform convergence on K. Then $\mathcal{U}$ is a base of neighborhoods for a (locally convex) topology, called the inductive limit topology, on $\mathcal{K}(X,\mathbb{C})$ [44]. This is the finest topology such that an injection $\mathcal{K}_K(X,\mathbb{C}) \to \mathcal{K}(X,\mathbb{C})$ is continuous. The inductive limit topology is finer than the topology of uniform convergence, and these topologies coincide if X is a compact space. Unless otherwise stated, referring to a topology on $\mathcal{K}(X,\mathbb{C})$, we will mean the inductive limit topology.

A complex measure on a locally compact space X is defined as a continuous form μ on a space $\mathcal{K}(X,\mathbb{C})$ of continuous complex functions of compact support on X. The value $\mu(f)$, $f \in \mathcal{K}(X,\mathbb{C})$, is called the integral $\int f\mu$ of f with respect to the measure μ. The space $M(X,\mathbb{C})$ of complex measures on X is the dual of $\mathcal{K}(X,\mathbb{C})$. It is provided with the weak* topology.

Given a complex measure μ, any continuous complex function $h \in \mathbb{C}^0(X)$ on X defines the continuous endomorphism $f \to hf$ of the space $\mathcal{K}(X,\mathbb{C})$ and yields a new complex measure $h\mu(f) = \mu(hf)$. Hence, the space $M(X,\mathbb{C})$ of complex measures on X is a module over the ring $\mathbb{C}^0(X)$.

Let $\mathcal{K}(X) \subset \mathcal{K}(X,\mathbb{C})$ denote a vector space of continuous real functions of compact support on X. The restriction of a complex measure μ on X to $\mathcal{K}(X)$ is a continuous complex form on $\mathcal{K}(X)$,

equipped with the inductive limit topology. Any complex measure μ is uniquely determined as

$$\mu(f) = \mu(\mathrm{Re}\, f) + i\mu(\mathrm{Im}\, f), \qquad f \in \mathcal{K}(X, \mathbb{C}), \tag{14.12}$$

by its restriction to $\mathcal{K}(X)$. A complex measure μ on X is called a real measure if its restriction to $\mathcal{K}(X)$ is a real form. A complex measure μ is real iff $\mu = \overline{\mu}$, where $\overline{\mu}$ is the conjugate measure given by the condition $\overline{\mu}(f) = \overline{\mu(\overline{f})}$, $f \in \mathcal{K}(X, \mathbb{C})$.

A measure on a locally compact space X is defined as a continuous real form on a space $\mathcal{K}(X)$ of continuous real functions of compact support on X provided with the inductive limit topology. Any real measure on $\mathcal{K}(X, \mathbb{C})$ restricted to $\mathcal{K}(X)$ is a measure. Conversely, each measure μ on $\mathcal{K}(X)$ is extended to the real measure (14.12) on $\mathcal{K}(X, \mathbb{C})$. Thus, measures on $\mathcal{K}(X)$ and real measures on $\mathcal{K}(X, \mathbb{C})$ can be identified.

A measure μ on a locally compact space X is called positive if $\mu(f) \geq 0$ for all positive functions $f \in \mathcal{K}(X)$. Any measure μ defines the positive measure $|\mu|(f) = |\mu(f)|$, and can be represented by the combination

$$\mu = \frac{1}{2}(|\mu| + \mu) - \frac{1}{2}(|\mu| - \mu)$$

of two positive measures.

A complex measure μ on a locally compact space X is called bounded if there is a positive number λ such that $|\mu(f)| \leq \lambda \|f\|$ for all $f \in \mathcal{K}(X, \mathbb{C})$. A complex measure μ is bounded iff it is continuous with respect to the topology of uniform convergence on $\mathcal{K}(X, \mathbb{C})$. Hence, a space $M^1(X, \mathbb{C}) \subset M(X, \mathbb{C})$ of bounded complex measures is the dual of $\mathcal{K}(X, \mathbb{C})$, provided with this topology. It is a Banach space with respect to a norm

$$\|\mu\| = \sup\{|\mu(f)| \,:\, \|f\| = 1\},\, f \in \mathcal{K}(X, \mathbb{C})\}. \tag{14.13}$$

Of course, any complex measure on a compact space is bounded. If μ is a bounded complex measure and h is a bounded continuous function on X, the complex measure $h\mu$ is bounded.

Similarly, a Banach space $M^1(X)$ of bounded measures on X is defined.

Example 14.3. Given a point $x \in X$, the assignment $\varepsilon_x : f \to f(x)$, $f \in \mathcal{K}(X)$, defines the Dirac measure on X. Any finite linear combination of Dirac measures is a measure, called a point measure. The Dirac measure ε_x is bounded, and $\|\varepsilon_x\| = 1$. □

Now we extend a class of integrable functions as follows. Let $\mathbb{R}_{\pm\infty}$ denote the extended real line, obtained from $\mathbb{R}$ by the adjunction of points $\{+\infty\}$ and $\{-\infty\}$. It is a ring such that $0 \cdot \infty = 0$ and $\infty - \infty = 0$. Let J_+ be a space of positive lower semicontinuous functions on X which take their values in the extended real line $\mathbb{R}_{\pm\infty}$. These functions possess the following important properties:

• the upper bound of any set of elements of J_+ and the lower bound of a finite set of elements of J_+ also are elements of J_+;

• any function $f \in J_+$ is an upper bound of a family of positive functions $h \in \mathcal{K}(X)$ such that $h \leq f$.

The last fact enables one to define the upper integral of a function $f \in J_+$ with respect to a positive measure μ on X as the element

$$\mu^*(f) = \int^* f\mu = \sup\{\mu(h) \,:\, h \in \mathcal{K}(X),\, 0 \leq h \leq f\} \tag{14.14}$$

of $\mathbb{R}_{\pm\infty}$. Of course, $\mu^*(f) = \mu(f)$ if $f \in \mathcal{K}(X)$.

Example 14.4. Let U be an open subset of X and φ_U its characteristic function. It is readily observed that $\varphi_U \in J_+$. Given a positive measure μ on X, the upper integral $\mu(U) = \mu^*(\varphi_U)$ is called the outer measure of U. For instance, the outer measure of a relatively compact set U (i.e., U is a subset of a compact set) is finite. The (finite or infinite) number $\mu(X) = \mu^*(1)$ is called the total mass of a measure μ. In particular, a measure on a locally compact space is bounded iff it has a finite total mass. $\Box$

Let f be an arbitrary positive $\mathbb{R}_{\pm\infty}$-valued function on a locally compact space X (not necessarily lower semicontinuous). There exist functions $g \in J_+$ such that $g \geq f$ (e.g., $g = +\infty$). Then the upper integral of f with respect to a positive measure μ on X is defined as

$$\mu^*(f) = \inf\{\mu^*(h) \,:\, h \in J_+,\, h \geq f\}. \tag{14.15}$$

Example 14.5. The outer measure $\mu^*(V) = \mu^*(\varphi_V)$ of an arbitrary subset V of X exemplifies the upper integral (14.15). In particular, one says that $V \subset X$ is a μ-null set if $\mu(V) = 0$. Two $\mathbb{R}_{\pm\infty}$-valued functions f and f' on a locally compact space are called μ-equivalent if they differ from each other only on a μ-null set; then $\mu^*(f) = \mu^*(f')$. Two positive measures μ and μ' are said to be equivalent if any compact μ-null set also is a μ'-null set, and *vice versa*. They coincide if $\mu(K) = \mu'(K)$ for any compact set $K \subset X$. $\Box$

Example 14.6. A real function f on a subset $V \subset X$ is said to be defined almost everywhere with respect to a positive measure μ on X if the complement $X \setminus V$ of V is a μ-null set. For instance, an $\mathbb{R}_{\pm\infty}$-valued function f which is finite almost everywhere on X exemplifies a real function defined almost everywhere on X. Conversely, one can think of a positive function defined almost everywhere on X as being μ-equivalent to some positive $\mathbb{R}_{\pm\infty}$-valued function on X. $\Box$

The following classes of integrable functions (and maps) are usually considered.

Let f be a map of a locally compact space X to a Banach space F, provided with a norm $|.|$ (e.g., F is $\mathbb{R}$ or $\mathbb{C}$). Given a positive measure μ on X, let us define the positive (finite or infinite) number

$$N_p(f) = \left[\int^* |f|^p\mu\right]^{1/p}, \qquad 1 \leq p < \infty. \tag{14.16}$$

Clearly, $N_p(f) = N_p(f')$ if f and f' are μ-equivalent maps on X, i.e., if they differ on a μ-null subset of X. There is the Minkowski inequality

$$N_p(f + f') \leq N_p(f) + N_p(f'). \tag{14.17}$$

- Let $R^p_F(X,\mu)$ be a space of maps $X \to F$ such that $N_p(f) < +\infty$. In accordance with the Minkowski inequality (14.17), it is a vector space and N_p (14.16) is a seminorm on $R^p_F(X,\mu)$. Provided with the corresponding topology, $R^p_F(X,\mu)$ is a complete space, but not necessarily Hausdorff. A space $\mathcal{K}(X,F)$ of continuous maps $X \to F$ of compact support belongs to $R^p_F(X,\mu)$.
- A space $\mathcal{L}^p_F(X,\mu)$ is defined as the closure of $\mathcal{K}(X,F) \subset R^p_F(X,\mu)$. Elements of $\mathcal{L}^p_F(X,\mu)$ are called integrable F-valued functions of degree p. In particular, elements of $\mathcal{L}^1_F(X,\mu)$ are called integrable F-valued functions, while those of $\mathcal{L}^2_F(X,\mu)$ are square integrable F-valued functions. Any element of $R^p_F(X,\mu)$ which is μ-equivalent to an element of $\mathcal{L}^p_F(X,\mu)$ belongs to $\mathcal{L}^p_F(X,\mu)$. An F-valued map defined almost everywhere on X also is called integrable if it is μ-equivalent to an element of $\mathcal{L}^p_F(X,\mu)$.

• A space $L^p_F(X,\mu)$ consists of classes of μ-equivalent integrable F-valued maps of degree p. One usually treat elements of this space as F-valued functions without fear of confusion, and call them integrable F-valued functions of degree p, too. The $L^p_F(X,\mu)$ is a Banach space with respect to the norm (14.16).

There are the following important relations between the spaces $L^p_F(X,\mu)$, $1\leq p<+\infty$.

If $f\in\mathcal{L}^p_F(X,\mu)$, then $|f|^{(p/q)-1}f$ belongs to $\mathcal{L}^q_F(X,\mu)$ for any $1\leq q<+\infty$, and *vice versa*. Moreover, $f\to|f|^{(1/q)-1}f$ provides a homeomorphism between topological spaces $\mathcal{L}^1_F(X,\mu)$ and $\mathcal{L}^q_F(X,\mu)$.

Let the numbers $1<p<+\infty$ and $1<q<+\infty$ obey the condition

$$p^{-1}+q^{-1}=1. \tag{14.18}$$

If $f\in L^p_{\mathbb{C}}(X,\mu)$ is an integrable complex function on X of degree p and $f'\in L^q_{\mathbb{C}}(X,\mu)$ is that of degree q, then $f\overline{f}'$ is integrable, i.e., belongs to $L^1_{\mathbb{C}}(X,\mu)$. In particular, a space $L^2_{\mathbb{C}}(X,\mu)$ of square integrable complex functions on a locally compact space X is a separable Hilbert space with respect to a scalar product

$$\langle f|f'\rangle=\int f\overline{f}'\mu. \tag{14.19}$$

One can say something more in the case of real functions. Let numbers p and q obey the condition (14.18). Any integrable real function $f\in\mathcal{L}^q_{\mathbb{R}}(X,\mu)$ on X of degree q defines a continuous real form

$$\phi_f: f'\to\int ff'\mu \tag{14.20}$$

on a space $L^p_{\mathbb{R}}(X,\mu)$ such that $N_q(f)=\|\phi_f\|$. Conversely, each continuous real form on $L^p_{\mathbb{R}}(X,\mu)$ is of type (14.20) where f is an element of $\mathcal{L}^q_{\mathbb{R}}(X,\mu)$ whose equivalence class in $L^q_{\mathbb{R}}(X,\mu)$ is uniquely defined. As a consequence, there is an isomorphism between Banach spaces $L^q_{\mathbb{R}}(X,\mu)$ and $(L^p_{\mathbb{R}}(X,\mu))'$, and a Banach space $L^p_{\mathbb{R}}(X,\mu)$ is reflexive.

Remark 14.7. One can define a space $L^\infty_{\mathbb{C}}(X,\mu)$ of complex infinite integrable functions on X as the dual of a Banach space $L^1_{\mathbb{C}}(X,\mu)$. In particular, any bounded continuous function belongs to $L^\infty_{\mathbb{C}}(X,\mu)$. Let us note that a space $L^1_{\mathbb{C}}(X,\mu)$ is not reflexive, i.e., the dual of $L^\infty_{\mathbb{C}}(X,\mu)$, provided with the strong topology, does not coincide with $L^1_{\mathbb{C}}(X,\mu)$. $\diamondsuit$

We now turn to the relation between equivalent measures. Let f be a positive $\mathbb{R}_{\pm\infty}$-valued function on a locally compact space X. Given a positive measure μ on X, a quantity

$$\widetilde{\mu}(f)=\sup_{K\subset X}\mu^*(\varphi_K f), \tag{14.21}$$

where K runs through a set of all compact subsets of X, is called the essential upper integral of f. Since $\varphi_K f\leq f$ for any compact subset K, the inequality $\widetilde{\mu}(f)\leq\mu^*(f)$ holds. In particular, if V is a subset of X and $\widetilde{\mu}(\varphi_V)=0$, one says that V is a locally μ-null set, i.e., any point $x\in X$ has a neighborhood U such that $U\cap V$ is a μ-null set. Essential upper integrals coincide with the upper ones if X is a locally compact space countable at infinity.

One says that a function on a subset V of X is defined locally almost everywhere if the complement of V is a locally null set. A real function f defined locally almost everywhere on X is called locally μ-integrable if any point $x\in X$ has a neighborhood U such that $\varphi_U f$ is a μ-integrable function or, equivalently, if hf is a μ-integrable function for any positive function $h\in\mathcal{K}(X)$ of compact support.

Let h be a locally μ-integrable function which is defined and non-negative almost everywhere on X. There is a positive measure $h\mu$ on X which obeys the relation $h\mu(f) = \mu(hf)$ for any $f \in \mathcal{K}(X)$. One says that $h\mu$ is a measure with the basis μ and the density h. For instance, if h is a μ-integrable function, then $h\mu$ is a bounded measure on X. This construction also is extended to complex functions and complex measures, seen as compositions of the positive real ones.

Theorem 14.1. Positive measures μ and μ' on a locally compact space X are equivalent iff $\mu' = f\mu$, where f is a locally μ-integrable function such that $f > 0$ locally almost everywhere on X. □

The function f in Theorem 14.1 is called the Radon–Nikodym derivative. Of course, $\widetilde{\mu}'(f') = \widetilde{\mu}(ff')$ for any positive integrable function f' on X.

14.4 Haar measures

Let us point out the peculiarities of measures on locally compact groups [9, 24].

Let G be a topological group acting continuously on a locally compact space X on the left, i.e., a map

$$\gamma(g) : X \ni x \to gx \in X, \qquad g \in G, \tag{14.22}$$

is continuous for any $g \in G$, and so is a map $G \ni g \to gx \in X$ for any $x \in X$. It should be emphasized that a map

$$G \times X \ni (g, x) \to gx \in X$$

need not be continuous.

Let f be a real function on X and μ a measure on X. A group G acts on f and μ by the laws

$$(\gamma(g)f)(x) = f(g^{-1}x), \qquad (\gamma(g)\mu)(f) = \mu(\gamma(g^{-1})f).$$

A measure $\gamma(g)\mu$ is the image of a measure μ with respect to the map (14.22).

A measure μ on X, subject to the action of a group G, is said to be:

- invariant if $\gamma(g)\mu = \mu$ for all $g \in G$;
- relative invariant, if there is a strictly positive number $\chi(g)$ such that $\gamma(g)\mu = \chi(g)^{-1}\mu$ for each $g \in G$;
- quasi-invariant, if measures μ and $\gamma(g)\mu$ are equivalent for all $g \in G$.

A strictly positive function $g \to \chi(g)$ yields a representation of G in $\mathbb{R}$. It is called the multiplier of a measure μ.

Let a topological group G act continuously on a locally compact space X on the right, i.e.,

$$\tau(g) : X \ni x \to xg^{-1} \in X.$$

The corresponding transformations of functions and measures on X read

$$(\tau(g)f)(x) = f(xg), \qquad (\tau(g)\mu)(f) = \mu(\tau(g^{-1})f).$$

Then invariant, relative invariant, and quasi-invariant measures on X are defined similarly to the case of G acting on X on the left.

Now let G be a locally compact group acting on itself by left and right multiplications

$$\begin{aligned} &\gamma(g) : q \to gq, \qquad \tau(g) : q \to qg^{-1}, \qquad q \in G, \\ &\gamma(g_1)\tau(g_2) = \tau(g_2)\gamma(g_1). \end{aligned} \tag{14.23}$$

Accordingly, left- and right-invariant measures, relative left- and right-invariant measures, left- and right-quasi-invariant measures on a group G are defined. Each measure μ on G also yields the inverse measure μ^{-1} given by a relation

$$\int f(g)\mu^{-1}(g) = \int f(g^{-1})\mu(g), \qquad f \in \mathcal{K}(G).$$

A positive (non-vanishing) left-invariant measure on a locally compact group G is called the left Haar measure (or, simply, the Haar measure). Similarly, the right Haar measure is defined.

Theorem 14.2. A locally compact group G admits a unique Haar measure with accuracy to a number multiplier. The total mass $\mu(G)$ of a Haar measure μ on G is finite iff G is a compact group. □

Let us choose, once and for all, a left Haar measure dg on a locally compact group G. If G is a compact group, dg is customarily the Haar measure of total mass 1.

Example 14.8. The Lebesgue measure dx is a Haar measure on the additive group $G = \mathbb{R}$. Its inverse is $-dx$. □

The equality (14.23) shows that, if dg is a Haar measure on G, a measure $\tau(g')dg$ for any $g' \in G$ also is left-invariant. Therefore, there exists a unique continuous strictly positive function $\Delta(g')$ on G such that $\tau(g')dg = \Delta(g')dg$, $g' \in G$. It is called the modular function of G. If dg is a left Haar measure, its inverse $(dg)^{-1}$ is a right Haar measure. There is a relation $(dg)^{-1} = \Delta(g)^{-1}dg$. If $\Delta(g) = 1$, a group G is called unimodular. Left and right Haar measures on a unimodular group differ from each other in a number multiplier. For instance, compact, commutative, and semisimple groups are unimodular. There is the following criterion of a unimodular group. If the unit element of a locally compact group has a compact neighborhood invariant under inner automorphisms, this group is unimodular.

Measures $\mu_1, \ldots, \mu_n$ on a locally compact group G are called mutually contractible if there exists a measure $\underset{i}{*}\mu_i$ on G given by a relation

$$\int f(g) \underset{i}{*}\mu_i(g) = \int f(g_1 \cdots g_n)\mu_1(g_1) \cdots \mu_n(g_n), \qquad f \in \mathcal{K}(G). \tag{14.24}$$

It is an image of the product measure $\mu_1 \cdots \mu_n$ on $\overset{n}{\times} G$ with respect to a map

$$\overset{n}{\times} G \ni (g_1, \ldots, g_n) \to g_1 \cdots g_n \in G.$$

Let ε_g, $g \in G$, be the Dirac measure on G. The following relations hold for all $x, y, z \in G$:

• $\varepsilon_x * \varepsilon_y = \varepsilon_{xy}$;

• $\varepsilon_x * \mu = \gamma(x)\mu$ and $\mu * \varepsilon_x = \tau(x^{-1})\mu$;

• if measures λ, μ, ν are contractible, the pairs of measures λ and μ, μ and ν, $\lambda * \mu$ and ν, λ and $\mu * \nu$ also are contractible, and we have

$$\lambda * \mu * \nu = (\lambda * \mu) * \nu = \lambda * (\mu * \nu).$$

For instance, any two bounded measures on G are contractible, and a space $M^1(G)$ of these measures is a unital Banach algebra with respect to the contraction $*$ (14.24), where the Dirac measure $\varepsilon_{\mathbf{1}}$ is the unit element.

One also defines:

• the contraction of a measure ν and a dg-integrable function f on G as the density of the contraction $\nu * (fdg)$ with respect to a Haar measure dg on G;

• the contraction of dg-integrable functions f_1 and f_2 on G as the density of the contraction $(f_1dg) * (f_2dg)$ with respect to a Haar measure dg on G, i.e.,

$$(f_1 * f_2)(g) = \int f_1(q)f_2(q^{-1}g)dq, \qquad q, g \in G. \tag{14.25}$$

14.5 Measures on infinite-dimensional vector spaces

Throughout this Section, E denotes a real Hausdorff topological vector space. Infinite-dimensional topological vector spaces need not be locally compact, and measures on them are defined as follows [9, 22, 24]. All measures are assumed to be positive.

Let $N(E)$ denote a set of closed vector subspaces of E of finite codimension, i.e., a vector subspace V of E belongs to $N(E)$ iff there exists a finite set $y_1, \ldots, y_n$ of elements of the dual E' of E such that V consists of $x \in E$ which obey the equalities $\langle x, y_i \rangle = 0$, $i = 1, \ldots, n$.

A quasi-measure (or a cylinder set measure in the terminology of [22]) on E is defined as family $\mu = \{\mu_V, V \in N(E)\}$ of bounded measures μ_V on finite-dimensional vector spaces E/V such that if $W \subset V$, the measure μ_V is the image of a measure μ_W with respect to the canonical morphism $E/W \to E/V$.

For instance, each bounded measure on E yields a quasi-measure $\{\mu_V, V \in N(E)\}$, where μ_V is the image of a measure μ with respect to the canonical morphism $r_V : E \to E/V$. There is one-to-one correspondence between the bounded measures on E and the quasi-measures on E which obey the following condition. For any $\varepsilon > 0$, there exists a compact subset $K \subset E$ such that

$$\mu_V(E/V - r_V(K)) \leq \varepsilon, \qquad V \in N(E).$$

Clearly, any quasi-measure on a finite-dimensional vector space is a measure.

Let $\gamma : E \to F$ be a continuous morphism of topological vector spaces. For any $W \in N(F)$, a subspace $V = \gamma^{-1}(W)$ of E belongs to $N(E)$, and γ yields a morphism $\gamma^W : E/V \to F/W$. Let $\mu = \{\mu_V, V \in N(E)\}$ be a quasi-measure on E. Then one can assign the measure

$$\nu_W = \gamma^W_*(\mu_{\gamma^{-1}(W)})$$

to each $W \in N(F)$. It is readily observed that the family $\nu = \{\nu_W, W \in N(F)\}$ is a quasi-measure on F. It is called the image of a quasi-measure μ with respect to a continuous morphism γ.

In particular, let $F = \mathbb{R}$, and let $y \in E'$ be a continuous form on E. The image μ_y of a quasi-measure μ on E with respect to a form y is a measure on $\mathbb{R}$. The Fourier transform of a quasi-measure μ on E is defined as a complex function

$$Z(y) = \int_{\mathbb{R}} e^{it}\mu_y(t) \tag{14.26}$$

on the dual E' of E. If μ is a bounded measure on E, its Fourier transform reads

$$Z(y) = \int_E \exp[i\langle x, y \rangle]\mu(x). \tag{14.27}$$

Let us point out the following variant of the well-known Bochner theorem. A complex function Z on a topological vector space F is called positive-definite if

$$\sum_{i,j} Z(y_i - y_j)\overline{c}_i c_j \geq 0$$

for any finite set $y_1, \ldots, y_m$ of elements of F and any complex numbers $c_1, \ldots, c_m$.

Theorem 14.3. The Fourier transform (14.27) provides a bijection of the set of quasi-measures on a Hausdorff topological vector space E to the set of positive-definite functions on the dual E' of E whose restriction to any finite-dimensional subspace of E' is continuous. □

For instance, let $M(y)$ be a seminorm on E'. Then a function

$$Z(y) = \exp\left[-\frac{1}{2}M(y)\right] \tag{14.28}$$

on E' is positive-definite. By virtue of Theorem 14.3, there is a unique quasi-measure μ_M on E whose Fourier transform is $Z(y)$ (14.28). It is called the Gaussian quasi-measure with a covariance form M.

Example 14.9. Let $E = \mathbb{R}^n$ be a finite-dimensional vector space, coordinated by (x^i), and let M be a norm on the dual of E. A Gaussian measure on E with a covariance form B is equivalent to the Lebesgue measure on E, and reads

$$\mu_M = \frac{\det[M]^{1/2}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(M^{-1})_{ij}x^i x^j\right] d^n x. \tag{14.29}$$

□

Example 14.10. Let E be a Banach space and E' its dual, provided with the norm $\|.\|'$ (14.1). A Gaussian quasi-measure on E with the covariance form $\|.\|'$ is called canonical. One can show that this quasi-measure fails to be a measure, unless E is finite-dimensional. Let T be a continuous operator in E'. Then

$$y \to \|Ty\| \tag{14.30}$$

is a seminorm on E'. A Gaussian quasi-measure on E with the covariance form (14.30) is proved to be a measure iff T is a Hilbert–Schmidt operator. □

Let E be a real nuclear space and E' its dual, equipped with the topology of uniform convergence. Let us recall that all topologies of uniform convergence (including weak* and strong topologies) on E' coincide, and E is reflexive. A quasi-measure on E' is a measure iff its Fourier transform on E (which is the dual of E') is continuous. A variant of the Bochner theorem for nuclear spaces states the following [22, 24].

Theorem 14.4. The Fourier transform

$$Z(x) = \int \exp[i\langle x, y\rangle]\mu(y)$$

provides a bijection of the set of measures on the dual E' of a real nuclear space E to the set of continuous positive-definite functions on E □

Remark 14.11. Let $E \subset \widetilde{E} \subset E'$ be a real rigged Hilbert space, defined by a norm $\|.\|$ on E. Let T be a nuclear operator in $\widetilde{E}$ and $\|.\|_T$ the restriction of the seminorm $y \to \|Ty\|$, $y \in \widetilde{E}$,

(14.30) on $\widetilde{E}$ to E. Then the Gaussian measures μ and μ_T on E' with the covariance forms $\|.\|$ and $\|.\|_T$ are not equivalent. The Gaussian measures μ and μ_T are equivalent if T is a sum of the identity and a nuclear operator. In particular, all Gaussian measures on a finite-dimensional vector space are equivalent. $\diamondsuit$

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